

# THE ALEXANDER POLYNOMIAL

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## 1. INTRODUCTION

The Alexander polynomial is a well understood classical knot invariant with interesting symmetry properties and recent applications in knot Floer homology [8, 7]. There are many different ways to compute the Alexander polynomial, some involving algebraic techniques and others more geometric or combinatorial approaches. This is an interesting example of how different types of mathematics can be used to describe the same result. While experts understand the relationships between different fields and methods of computation, the subtleties are often omitted in the literature. This paper describes four routes to the Alexander polynomial with the intent to explicate these subtleties and bring clarity to this intersection of subjects.

The format of this paper is first to describe Alexander's original definition of the Alexander polynomial, a purely algebraic definition. Then a detailed construction of Seifert surfaces and the infinite cyclic cover of a knot complement will be given. Next, we will show that the Alexander polynomial can be calculated using a Seifert Matrix, which is still an algebraic result but has a more geometric flavor. While Seifert surfaces have rich content useful as a basis for proofs, perhaps the easiest way to compute the Alexander polynomial is using Conway's method of Skein relations. This method is very combinatorial and surprisingly simple. The next section of the paper proves that the Alexander polynomial is the unique knot invariant satisfying the Skein relations and initial condition. Lastly, a brief introduction to grid diagrams will lead to the final description of the Alexander polynomial via the Minesweeper matrix.

Notation:  $\Lambda = \mathbb{Z} \langle t \rangle$  is the ring of Laurent polynomials.

$\Delta_K = \Delta(K)$  is the Alexander polynomial for  $K$ .

All knots referred to in this paper are assumed to be tame knots.

### 1.1. Background.

**Definition 1.1.** A **knot** is an embedding of  $S^1$  in  $\mathbb{R}^3$  (or in  $S^3$ ).

**Definition 1.2.** An **ambient isotopy** is a continuous map from  $[0, 1] \times \mathbb{R}^3$  onto  $\mathbb{R}^3$  that is a homeomorphism at every level.

**Definition 1.3.** Two knots are **equivalent** if there exists an ambient isotopy mapping one knot onto the other.

**Definition 1.4.** A **knot invariant** is a function on the set of all knots that assigns equivalent knots the same output.

**Definition 1.5.** The **knot group** of a knot is the fundamental group of the knot complement in  $\mathbb{R}^3$ , or in  $S^3$ .

**Theorem 1.6.** (*Wirtinger Presentation*, [5]) *The knot group for any knot has a finite presentation  $(b_1, \dots, b_k : r_1, \dots, r_k)$  where each relator  $r_i$  is of the form  $b_j b_l b_n^{-1} b_l^{-1}$ .*

The Wirtinger presentation arises by taking loops that pass once under each strand of the knot as generators and the relators occur at the crossings.

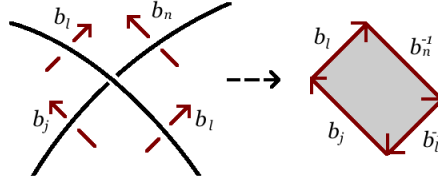


FIGURE 1. Wirtinger crossing relations

## 2. ALEXANDER'S ORIGINAL DEFINITION THROUGH THE ALEXANDER MODULE

James Waddell Alexander II developed the Alexander polynomial in 1923, which was the first knot invariant in the form of a polynomial. While this polynomial has surprising qualities, Alexander's original development is quite natural. Defining knot equivalence using ambient isotopies immediately gives the knot group as a knot invariant. This section will define the Alexander module as a specific quotient space of derived subgroups of the knot group and construct a knot invariant from this space.

Let  $K$  be a knot and  $X = S^3 - K$ . Denote  $G = \pi_1(X)$ .

**Proposition 2.1.**  $H_1(X) \cong \langle t \rangle$ .

*Proof.* A Wirtinger presentation of  $G$  is  $(b_1, \dots, b_k : r_1, \dots, r_{k-1})$  where each  $r_i$  is of the form  $b_j b_l b_n^{-1} b_l^{-1}$ . So there exists a homomorphism  $\varphi : G \rightarrow \langle t \rangle$ , where  $\langle t \rangle$  is the infinite cyclic free abelian group, given by  $\varphi(b_i) = t$ . This map is well defined on  $G$  since  $\varphi(r_i) = 1$  for each relator  $r_i$ . Now  $G'$  is the derived subgroup of  $G$ . Since  $\langle t \rangle$  is abelian, then  $\varphi$  factors thru to induce a map  $\bar{\varphi}$  on  $G/G'$ .

The relator  $b_j b_l b_n^{-1} b_l^{-1}$  gives that  $b_j$  is conjugate to  $b_n$ . Following the relators around the entire knot gives that each generator is conjugate to every other generator. Conjugacy classes are preserved by homomorphisms, so the image of  $\bar{\varphi}$  is generated by some  $b_1 G'$ . So  $\bar{\varphi}(b_1 G') = t$ . This mapping is an isomorphism by existence of the inverse function  $t^i \rightarrow b_1^i G'$ . Hence  $G/G' \cong \langle t \rangle$ .

By the Hurewicz theorem,  $H_1(S^3 - K) \cong \pi_1(S^3 - K)/[\pi_1(S^3 - K), \pi_1(S^3 - K)]$ , So  $H_1(S^3 - K) \cong \langle t \rangle$ .  $\square$

**Proposition 2.2.** *For any knots  $K$  and  $L$ , denote  $G = \pi_1(S^3 - K)$  and  $H = \pi_1(S^3 - L)$ . If  $G \cong H$  then  $G'/G'' \cong H'/H''$ . Moreover,  $G'/G''$  is an invariant of the knot group.*

*Proof.* Let  $f : G \rightarrow H$  be an isomorphism. If  $xyx^{-1}y^{-1} \in G'$  then  $f(xyx^{-1}y^{-1}) = f(x)f(y)f(x)^{-1}f(y)^{-1} \in H'$ . Similarly, if  $zwz^{-1}w^{-1} \in H'$  then  $f^{-1}(zwz^{-1}w^{-1}) = f^{-1}(z)f^{-1}(w)f^{-1}(z^{-1})f^{-1}(w^{-1}) \in G'$ . So  $f(G') = H'$  giving  $G' \cong H'$ . A similar proof gives that  $G'' \cong H''$ , thus giving that  $G'/G'' \cong H'/H''$ .  $\square$

**Corollary 2.3.**  *$G'/G''$  is a knot invariant.*

*Proof.*  $G'/G''$  is an invariant of the knot group, which is a knot invariant.  $\square$

**Proposition 2.4.**  *$G'/G''$  is a  $\Lambda$ -module.*

*Proof.* By Proposition 2.1,  $G/G' \cong \langle t \rangle$ . This structure is used to define a  $\Lambda$  action on  $G'/G''$ . Let  $g \in G$  such that  $gG' = t^k$ . For  $\gamma \in G'$  define  $t^k \gamma = g\gamma g^{-1} \in G'$ . However, if  $hG' = t^k$  then  $t^k \gamma = h\gamma h^{-1}$ . Since  $gG' = hG'$  then  $g^{-1}h \in G'$ . So,

$$\begin{aligned} g\gamma g^{-1}(h\gamma h^{-1})^{-1} &= g\gamma g^{-1}(h\gamma^{-1}h^{-1}) \\ &= g\gamma g^{-1}(h\gamma^{-1}h^{-1})gg^{-1} \\ &= g\gamma g^{-1}h\gamma^{-1}(g^{-1}h)^{-1}g^{-1} \in G'' \end{aligned}$$

Hence,  $g\gamma g^{-1}G'' = h\gamma h^{-1}G''$ , giving a well defined  $\Lambda$  action on  $G'/G''$ .  $\square$

**Definition 2.5.** The **Alexander module** of a knot  $K$  is  $G'/G''$ , denoted  $A_K$ .

**Definition 2.6.** Let  $M = (\alpha_1, \dots, \alpha_t : r_1, \dots, r_s)$  be a finitely presented Module over ring  $R$ . A **presentation matrix** for  $M$  corresponding to the given presentation is  $P = [p_{ij}]$  where  $r_i = \sum_{k=1}^t p_{ik} \alpha_k$ , for some  $p_{ij} \in R$ .

**Definition 2.7.** Let  $M$  be a module over  $R$  with an  $s \times r$  presentation matrix  $P$ , the **Order Ideal** of  $M$  is the ideal in  $R$  generated by the  $r \times r$  minor matrices of  $P$ . If  $s < r$  then the order ideal of  $M$  is the zero ideal.

**Definition 2.8.** The **Alexander polynomial** of  $K$ , denoted  $\Delta_K$ , is a generator of the order ideal of a presentation matrix for the Alexander module.

**Note 2.9.**  $\Delta_k$  is defined up to multiplication by a unit in  $\Lambda$ , i.e. multiplication by  $\pm t^n$ .

It will later be shown, in Theorem 3.11, that the Alexander module has a square presentation matrix. The order ideal of this presentation matrix is principle, thus the Alexander polynomial is well defined.

**Theorem 2.10.** *The Alexander polynomial is a knot invariant.*

To prove this theorem, the following definition and theorem are required.

**Definition 2.11.** [5] Two matrices are **equivalent** if one can be obtained from the other by a finite sequence of the following operations and their inverse operations:

- (1) Permute rows or columns
- (2) Adjoin a new row of zeroes
- (3) Add a multiple of a row (or column) to any other row (or column)
- (4) Multiply a row (or column) by a unit
- (5) Replace the  $m \times n$  matrix  $M$  by the *bordered*  $(m+1) \times (n+1)$

$$\text{matrix : } \begin{pmatrix} M & 0 \\ & \vdots \\ 0 \cdots 0 & 1 \end{pmatrix}.$$

**Theorem 2.12.** ([5], [13]) *Any two presentation matrices for a finitely presented module are equivalent.*

*Proof of Theorem 2.10.* The Alexander polynomial is defined as a generator of the order ideal of a presentation matrix of the Alexander

module. Corollary 2.3 shows that the Alexander module is a knot invariant. So, it remains to show that the Alexander polynomial is an invariant of the Alexander module.

By Theorem 2.12, to show the Alexander polynomial is an invariant of the Alexander module  $A_K$ , it suffices to show that the same order ideal is generated by equivalent presentation matrices for  $A_K$ . To do this, consider the effect of the operations in definition 2.11 on the square minor determinants of equivalent matrices.

- (1) Permuting rows or columns changes the sign of the determinant.
- (2) Adjoining a row of zeroes creates new minor matrices with zero determinants.
- (3) Adding a multiple of a row (or column) to any other row (or column) does not change the determinant.
- (4) All units in  $\Lambda$  are of the form  $\pm t^k$  for  $k \in \mathbb{Z}$ . Multiplying a row or column by  $\pm t^k$  corresponds to multiplying the determinant by  $\pm t^k$ .
- (5) Bordering a matrix  $M$  creates new minor matrices with zero determinant and the minors  $1 \cdot \det(\tilde{M})$  for  $\tilde{M}$  minors of  $M$ .

The same order ideals will be generated by equivalent matrices since the minor determinants are only changed by  $\pm 1$  or  $\pm t^k$ , which are units in  $\Lambda$ . Therefore the Alexander polynomial is completely determined by equivalent matrices, and hence is an invariant of  $A_K$ .  $\square$

### 3. PRESENTATION MATRICES

One difficulty with Alexander's original definition is that derived subgroups are generally complicated and hard to describe with a finite presentation. This leaves the Alexander module and Alexander polynomial as intangible abstract objects. However, if  $X$  is a knot complement, with fundamental group  $G$ , in Proposition 2.1, it was shown that  $H_1(X) = \langle t \rangle$ . So from a topological standpoint, if  $\tilde{X}$  is the maximal abelian cover of  $X$  then  $H_1(\tilde{X}) \cong G'/G''$  is the Alexander module. Moreover,  $\tilde{X}$  is the cover defined by  $p_*(\pi_1(\tilde{X}, \tilde{x})) = G' \trianglelefteq \pi_1(X, x)$ . Thus  $\pi_1(\tilde{X}) \cong G'$  and  $H_1(\tilde{X}) \cong G'/G''$ . So, the homology of a maximal abelian cover of the knot complement gives an interpretation of the Alexander module. This turns out to be a lucrative approach and gives rise to our second description of the Alexander polynomial via the Seifert matrix. This next section develops the Seifert matrix using Seifert surfaces and a description of  $\tilde{X}$  as an infinite cyclic cover of the knot complement.

#### 3.1. Seifert Surfaces.

**Definition 3.1.** A subset  $X \subset Y$  is said to be **bicollared** in  $Y$  if there exists an embedding  $h : X \times [-1, 1] \rightarrow Y$  such that  $h(x, 0) = x$  when  $x \in X$ . The map or its image is then said to be the bicollar.

**Definition 3.2.** A **Seifert Surface** for a knot, or link,  $K \subset S^3$  is a connected, bicollared, compact manifold  $M \subset S^3$  with  $\partial M = K$ .

**Note 3.3.** A bicollared surface is always orientable as a subset of  $S^3$ .

**Theorem 3.4.** *Every knot and link admits a Seifert surface.*

*Proof.* Fix an orientation of the knot or link. Smooth out the crossings by following Seifert's algorithm shown in Figure 2. After this smoothing, the diagram will be several simple closed oriented curves called Seifert Circles.

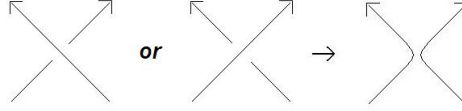


FIGURE 2. crossing smoothing

Individually, each Seifert circle bounds a disk the plane. If any Seifert circles are nested, pull the circles out of the plane into a three dimensional "stack". Thus the Seifert circles are disjoint, oriented and each bounds a disk in  $\mathbb{R}^3$ . Each disk is bicollared and can be assigned a  $+$  and  $-$  side by the convention that the  $+$  side has a counterclockwise boundary orientation. To create the surface, attach a half-twisted strip to each position where a crossing was smoothed. The half twist crosses in the same manner of the original crossing, leaving the boundary of the surface to be the original knot, or link.

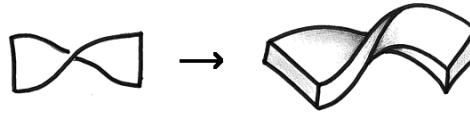


FIGURE 3. Half-twisted strip with and without the bicollar

An example construction of a Seifert surface for the figure 8 knot is shown in Figure 4. This surface is orientable and comes equipped with a bicollar, but the surface drawn does not show the bicollar.

Now, if the link is disconnected, then following this algorithm will leave a disconnected surface. To remedy this, connect the disjoint surfaces by a hollow pipe that preserves the bicollar but does not change the boundary of the surface, see Figure 5.

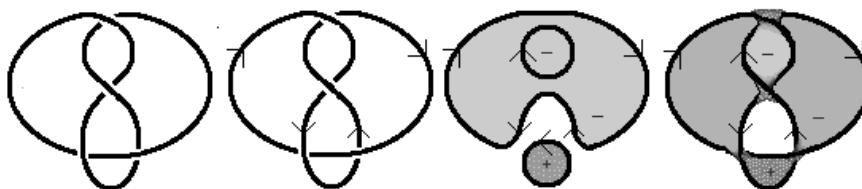


FIGURE 4. Construction of a Seifert surface

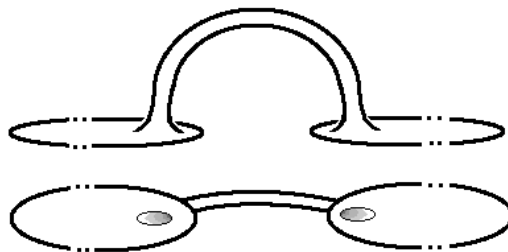


FIGURE 5. Connecting disjoint surfaces with a hollow pipe.

□

**Note 3.5.** A Seifert surface depends on the choice of orientation and projection of the knot.

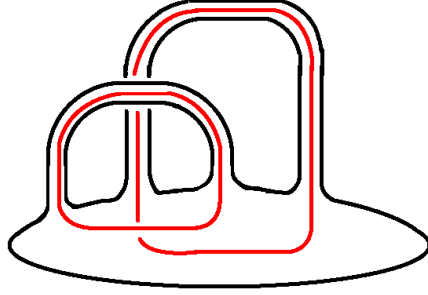
As a consequence of this construction, we can define the **genus of a knot** to be the minimum genus of any Seifert surface for the knot. This is a knot invariant that is easily defined, but difficult to compute. Interestingly, the Alexander polynomial gives a nice result about the genus of a knot in Corollary 3.29.

*Remark 3.6.* All punctured compact orientable manifolds with connected boundary and genus  $g$  are homeomorphic to a disk with  $2g$  "handles" attached. These handles may be twisted and intertwined. Let  $M$  be such a surface. Up to homotopy type,  $\overset{\circ}{M}$  is a wedge of  $2g$  circles,  $\overset{\circ}{M} \cong \vee_{i=1}^{2g} S^1$ . So  $H_1(\overset{\circ}{M}) = H_1(\vee_{i=1}^{2g} S^1) = \mathbb{Z}^{2g}$ . Then a basis for  $H_1(\overset{\circ}{M})$  can be found by collecting the  $2g$  loops that pass through the disk and one handle, see Figure 6.

### 3.2. Seifert Matrix

The following basic properties of Homology theory are accepted without proof.

For any topological space  $W$ ,

FIGURE 6. Generators for  $H_1(\overset{\circ}{M})$ 

- Each loop  $f : S^1 \rightarrow W$  represents an element  $[f] \in H_1(W)$ .
- For every  $a \in H_1(W)$ , there exists a loop  $f : S^1 \rightarrow W$  such that  $[f] = a \in H_1(W)$ .
- If two loops  $f, g : S^1 \rightarrow W$  are freely homotopic, then  $[f] = [g] \in H_1(W)$ .

**Definition 3.7.** Let  $J$  and  $K$  be two disjoint oriented knots, or links. For each point where  $J$  crosses *under*  $K$  in the projection of  $J$  and  $K$ , assign a  $\pm 1$  following Figure 7. Then the linking number of  $J$  and  $K$ , denoted  $\text{lk}(J, K)$ , is the sum of the assigned numbers over all crossings.

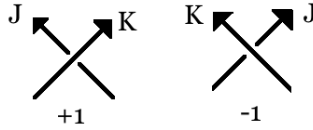


FIGURE 7. Crossing assignment for linking number.

We may extend this definition to homology classes. That is, if  $[J], [K] \in H_1(W)$ , then  $\text{lk}([J], [K]) = \text{lk}(J, K)$ .

**Proposition 3.8.** ([10] pg 135-136) *The linking number has the following properties:*

- (1) *If there are homotopies  $J_t : S^1 \rightarrow \mathbb{R}^3$  and  $K_t : S^1 \rightarrow \mathbb{R}^3$  such that  $\text{Im}(J_t) \cap \text{Im}(K_t) = \emptyset$  for all  $t \in [0, 1]$  then  $\text{lk}(J_0, K_0) = \text{lk}(J_1, K_1)$ .*
- (2)  $\text{lk}(J, K) = \text{lk}(K, J)$ .
- (3)  $\text{lk}(J, K) = [J] \in H_1(S^3 - K)$
- (4)  *$\text{lk}$  is bilinear, in the sense that  $\text{lk}([J] + [J'], [K]) = \text{lk}([J], [K]) + \text{lk}([J'], [K])$ .*

For the following definitions, let  $\overset{\circ}{M}$  be the interior of a Seifert surface for a knot, and  $\mathcal{N} : \overset{\circ}{M} \times (-1, 1) \rightarrow S^3$  an open bicollar for  $M$ .



**Definition 3.9.** A *Seifert Form* is a function  $f : H_1(\overset{\circ}{M}) \times H_1(\overset{\circ}{M}) \rightarrow \mathbb{Z}$  given by  $f([x], [y]) = lk(x, \mathcal{N}(y, \frac{1}{2}))$ .

Note that Definition 3.9 is well defined by Proposition 3.8 (1).

**Definition 3.10.** A *Seifert Matrix* is a  $2g \times 2g$  matrix  $V_{ij} = f(e_i, e_j)$  where  $e_i$ 's are basis for  $H_1(\overset{\circ}{M})$ ,  $f$  is a Seifert form and  $g$  is the genus of the Seifert Surface  $M$ .

**Theorem 3.11.** If  $V$  is the Seifert matrix for a knot  $k$  in  $S^3$ , then  $V^T - tV$  is a presentation matrix for  $A_k$ .

The Seifert surface, and ultimately the Seifert matrix, is a geometric object. Theorem 3.11 relates the Alexander polynomial, a purely algebraic object, with a geometric object. In order to prove this theorem we must first understand the algebraic structure of the Seifert surface. Finally to do this, we need to understand the algebraic structure of the infinite cyclic cover of the knot complement. The next section describes this process.

### 3.3. Infinite Cyclic Cover of a Knot Complement.

This construction follows Rolfsen [10].

The infinite cyclic cover of a knot complement is a complicated space whose first homology group is the Alexander module of the knot. To give better intuition of the the space, this detailed construction will be accompanied by schematic diagrams.

Let  $M$  be a Seifert Surface for knot  $K$ . By definition,  $M$  is bicollared, so there exists an open bicollar  $\mathcal{N} : \overset{\circ}{M} \times (-1, 1) \rightarrow S^3$ , where  $\overset{\circ}{M} = M - K$  the interior of  $M$ . Thus  $\overset{\circ}{M} = \mathcal{N}(\overset{\circ}{M} \times 0)$ . We use the following notation:

$$N = \mathcal{N}(\overset{\circ}{M} \times (-1, 1)).$$

$$N^+ = \mathcal{N}(\overset{\circ}{M} \times (0, 1)), \text{ the positive side of } N.$$

$$N^- = \mathcal{N}(\overset{\circ}{M} \times (-1, 0)), \text{ the negative side of } N.$$

$$N^\pm = N^- \dot{\cup} N^+.$$

$$Y = S^3 - M.$$

$$X = S^3 - K.$$

Notice that  $N \cap Y = N^\pm$  and that  $X$  is the union of two open sets  $Y$  and  $N$ .

An infinite cyclic cover  $\tilde{X}$  for  $X$  is constructed as follows. Form two disjoint unions

$$Y \times \mathbb{Z} = \dot{\cup}_{k \in \mathbb{Z}} Y_k$$

$$N \times \mathbb{Z} = \dot{\bigcup}_{k \in \mathbb{Z}} N_k$$

where  $Y_k = Y \times \{k\}$  is a copy of  $Y$  and similarly for  $N_k$ . Here,  $\mathbb{Z}$  has the discrete topology.

The inclusion of  $N^\pm$  into  $N$  naturally extends to an inclusion  $k_N : N^\pm \times \mathbb{Z} \rightarrow N \times \mathbb{Z}$ . We also have the inclusion of  $N^\pm$  into  $Y$ . Define  $b : N^\pm \times \mathbb{Z} \rightarrow Y \times \mathbb{Z}$  by

$$b(x, k) = \begin{cases} (x, k+1) & x \in N^+ \\ (x, k) & x \in N^- \end{cases}$$

The map  $b$  is continuous because  $N^\pm$  has the discrete topology.

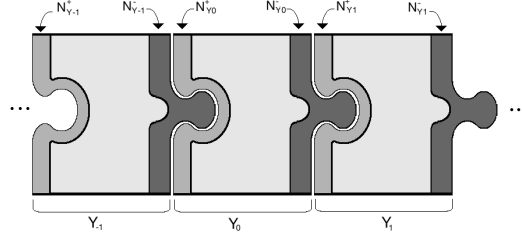


FIGURE 8. Schematic diagram of  $Y_i$ 's. The "gap" between the  $Y_i$ 's is a copy of  $\mathring{M}$ .

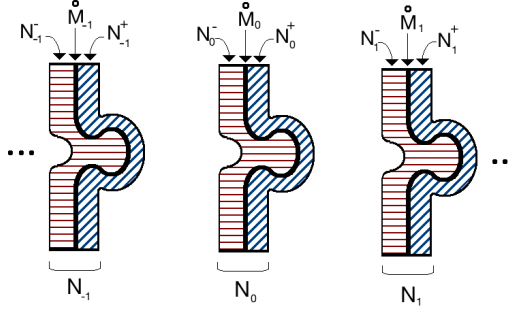


FIGURE 9. Schematic diagram of  $N_i$ 's

So the  $N_i^\pm$ 's are like pieces of tape used to paste together copies of the complement of the surface and copies of the surface without the knot.

Define  $\tilde{X}$  as the push out as in Figure 11. This induces maps  $j_Y : Y \times \mathbb{Z} \rightarrow \tilde{X}$  and  $j_N : N \times \mathbb{Z} \rightarrow \tilde{X}$  and the push out has the following universal properties:

$$(1) \quad j_Y \circ b = j_N \circ k_N$$

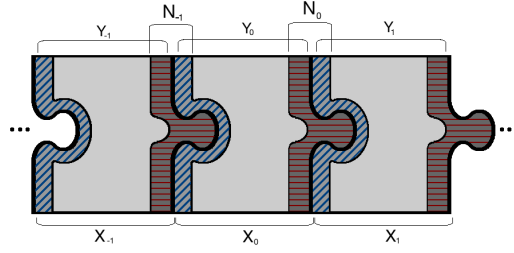


FIGURE 10. Schematic diagram of  $\tilde{X}$ . The "gap" between the  $Y_i$ 's is filled by the copies of  $\tilde{M}$  in the  $N_i$ 's, leaving a path connected space.

$$\begin{array}{ccc}
 (N^- \dot{\cup} N^+) \times \mathbb{Z} & \xrightarrow{b} & Y \times \mathbb{Z} \\
 k_N \downarrow & \curvearrowright & \downarrow j_Y \\
 N \times \mathbb{Z} & \xrightarrow{j_N} & \tilde{X}
 \end{array}$$

FIGURE 11

- (2) For any space  $V$  and maps  $g : N \times \mathbb{Z} \rightarrow V$  and  $f : Y \times \mathbb{Z} \rightarrow V$  such that  $b \circ f = k_N \circ g$ , there exists a unique induced map  $h : \tilde{X} \rightarrow V$  with the property that  $h \circ j_Y = f$  and  $h \circ j_N = g$ .

$$\begin{array}{ccccc}
 N \times \mathbb{Z} & \xrightarrow{j_N} & \tilde{X} & \xleftarrow{j_Y} & Y \times \mathbb{Z} \\
 & \searrow g & \downarrow h & \swarrow f & \\
 & & V & & 
 \end{array}$$

FIGURE 12

The space  $\tilde{X}$  is constructed as a quotient space of  $(N \times \mathbb{Z}) \dot{\cup} (Y \times \mathbb{Z})$  with the equivalence relation that  $k_N(x, k) \sim b(x, k)$  for all  $(x, k) \in (N^- \dot{\cup} N^+) \times \mathbb{Z}$ . There are more natural inclusion maps  $h_N : N \rightarrow \tilde{X}$ ,  $h_Y : Y \rightarrow \tilde{X}$  and projection maps  $p_N : N \times \mathbb{Z} \rightarrow N$  and  $p_Y : Y \times \mathbb{Z} \rightarrow Y$ . Figure 13 is a diagram relating these maps.

**Proposition 3.12.** *The map  $j_Y$  embeds  $Y \times \mathbb{Z}$  as an open subset of  $\tilde{X}$ . The map  $j_N$  embeds  $N \times \mathbb{Z}$  as an open subset of  $\tilde{X}$ .*

*Proof.* Let  $\pi$  be the quotient projection from  $U := (N \times \mathbb{Z}) \dot{\cup} (Y \times \mathbb{Z})$  onto  $\tilde{X}$ . It suffices to show that both  $\pi^{-1} \circ j_Y(Y \times \mathbb{Z})$  and  $\pi^{-1} \circ j_N(N \times \mathbb{Z})$  are open in  $U$ .

First note that  $j_N$  and  $j_Y$  are open maps by the construction of  $\tilde{X}$ . Now  $N^\pm$  is an open subset of  $N$ , so  $k_N$  embeds  $N^\pm \times \mathbb{Z}$  as an open set in  $N \times \mathbb{Z}$ . Similarly,  $b$  embeds  $N^+ \times \{k\}$  in  $Y \times \{k+1\}$  and  $N^- \times \{k\}$  in  $Y \times \{k\}$ . Since  $N^\pm$  is an open subset of  $Y$ , then  $b$  embeds  $N^\pm \times \mathbb{Z}$  as an open set in  $Y \times \mathbb{Z}$ . So,

$$\begin{aligned}\pi^{-1} \circ j_N(N \times \mathbb{Z}) &= b(N^\pm \times \mathbb{Z}) \dot{\cup} (N \times \mathbb{Z}) \stackrel{op}{\subset} U \\ \pi^{-1} \circ j_Y(Y \times \mathbb{Z}) &= k_N(N^\pm \times \mathbb{Z}) \dot{\cup} (Y \times \mathbb{Z}) \stackrel{op}{\subset} U.\end{aligned}$$

□

**Proposition 3.13.** *The maps  $p_Y \circ h_Y$  and  $p_N \circ h_N$  induce a map  $p : \tilde{X} \rightarrow X$ .*

$$\begin{array}{ccccc} (N^- \dot{\cup} N^+) \times \mathbb{Z} & \xrightarrow{b} & Y \times \mathbb{Z} & \xrightarrow{p_Y} & Y \\ \downarrow k_N & & \downarrow j_Y & & \downarrow h_Y \\ N \times \mathbb{Z} & \xrightarrow{j_N} & \tilde{X} & \xrightarrow{p} & X \\ \downarrow p_N & & & & \downarrow h_N \\ N & \hookrightarrow & & & X \end{array}$$

FIGURE 13

*Proof.* By the universal property of the push out  $\tilde{X}$ , it suffices to show that  $h_Y \circ p_Y \circ b = h_N \circ p_N \circ k_N$ . Consider cases on  $(x, k) \in (N^- \dot{\cup} N^+) \times \mathbb{Z}$ .

- Case 1:  $x \in N^+$ . Then

$$\begin{aligned}h_Y \circ p_Y \circ b(x, k) &= h_Y \circ p_Y(x, k+1) = h_Y(x) = x \\ &= h_N(x) = h_N \circ p_N(x, k) = h_N \circ p_N \circ k_N(x, k).\end{aligned}$$

- Case 2:  $x \in N^-$ . Then clearly

$$\begin{aligned}h_Y \circ p_Y \circ b(x, k) &= h_Y \circ p_Y(x, k) = h_Y(x) = x \\ &= h_N(x) = h_N \circ p_N(x, k) = h_N \circ p_N \circ k_N(x, k).\end{aligned}$$

So  $h_Y \circ p_Y \circ b(x, k) = h_N \circ p_N \circ k_N(x, k)$  for all  $(x, k) \in (N^- \dot{\cup} N^+) \times \mathbb{Z}$ .

□

Next, define shifting maps  $s_N : N \times \mathbb{Z} \rightarrow N \times \mathbb{Z}$  and  $s_Y : Y \times \mathbb{Z} \rightarrow Y \times \mathbb{Z}$  by  $s_N(x, k) = (x, k+1)$  and  $s_Y(x, k) = (x, k+1)$ .

**Proposition 3.14.** *The maps  $j_N \circ s_N$  and  $j_Y \circ s_Y$  induce a map  $\tau : \tilde{X} \rightarrow \tilde{X}$ .*

$$\begin{array}{ccccc}
(N^- \dot{\cup} N^+) \times \mathbb{Z} & \xrightarrow{b} & Y \times \mathbb{Z} & \xrightarrow{s_Y} & Y \times \mathbb{Z} \\
\downarrow k_N & & \downarrow j_Y & & \downarrow j_Y \\
N \times \mathbb{Z} & \xrightarrow{j_N} & \tilde{X} & \cdots & \tilde{X} \\
\downarrow s_N & & \downarrow \tau & & \downarrow \tau \\
N \times \mathbb{Z} & \xrightarrow{j_N} & \tilde{X} & & \tilde{X}
\end{array}$$

FIGURE 14

*Proof.* By the universal property of the push out  $\tilde{X}$ , it suffices to show that  $j_N \circ s_N \circ k_N = j_Y \circ s_Y \circ b$ . Consider cases on  $(x, k) \in (N^- \dot{\cup} N^+) \times \mathbb{Z}$ .

- Case 1:  $x \in N^+$ . Then

$$\begin{aligned}
j_Y \circ s_Y \circ b(x, k) &= j_Y \circ s_Y(x, k+1) = j_Y(x, k+2) = j_Y \circ b(x, k+1) \\
&= j_N \circ k_N(x, k+1) = j_N(x, k+1) = j_N \circ s_N(x, k) \\
&= j_N \circ s_N \circ k_N(x, k).
\end{aligned}$$

- Case 2:  $x \in N^-$ . Then similarly

$$\begin{aligned}
j_Y \circ s_Y \circ b(x, k) &= j_Y \circ s_Y(x, k) = j_Y(x, k+1) = j_Y \circ b(x, k+1) \\
&= j_N \circ s_N(x, k) = j_N \circ s_N \circ k_N(x, k)
\end{aligned}$$

So  $j_N \circ s_N \circ k_N(x, k) = j_Y \circ s_Y \circ b(x, k)$  for all  $(x, k) \in (N^- \dot{\cup} N^+) \times \mathbb{Z}$ .  $\square$

*Remark 3.15.* A similar construction using shifting maps that reduce the indices by 1 would induce a map  $\tau^{-1} : \tilde{X} \rightarrow \tilde{X}$  such that  $\tau^{-1}(x, k) = (x, k-1)$ . Thus  $\tau$  is a homeomorphism on  $\tilde{X}$ .

**Proposition 3.16.**  $p \circ \tau = p$ .

*Proof.* By definition,  $p$  is the *unique* induced map such that  $p \circ j_Y = h_Y \circ p_Y$  and  $p \circ j_N = h_N \circ p_N$ . So, it suffices to show that  $p \circ \tau \circ j_Y = h_Y \circ p_Y$  and  $p \circ \tau \circ j_N = h_N \circ p_N$ .

$$\begin{aligned}
p \circ \tau \circ j_N &= p \circ j_N \circ s_N && \text{(by definition of } \tau \text{)} \\
&= h_N \circ p_N \circ s_N && \text{(by definition of } p \text{)} \\
&= h_N \circ p_N && \text{(because } p_N = p_N \circ s_N \text{)}
\end{aligned}$$

Similarly,

$$\begin{aligned}
p \circ \tau \circ j_Y &= p \circ j_Y \circ s_Y && \text{(by definition of } \tau \text{)} \\
&= h_Y \circ p_Y \circ s_Y && \text{(by definition of } p \text{)} \\
&= h_Y \circ p_Y && \text{(because } p_Y = p_Y \circ s_Y \text{)}
\end{aligned}$$

□

**Lemma 3.17.**  *$p$  is a covering projection.*

*Proof.* Since  $X = N \cup Y$ , it suffices to show that  $N$  and  $Y$  are evenly covered by  $p$ .

$$\begin{aligned} p^{-1}(N) &= j_N \circ p_N^{-1} \circ h_N^{-1}(N) = j_N \circ p_N^{-1}(N) \\ &= j_N(N \times \mathbb{Z}). \end{aligned}$$

Proposition 3.12, proved that  $j_N(N \times \mathbb{Z})$  is an embedded subset of  $\tilde{X}$ .

Claim: For a fixed  $k$ ,  $p|_{j_N(N \times \{k\})}$  is a homeomorphism onto  $N$ .

Let  $S : N \rightarrow N \times \{k\}$  be the slice map  $S(n) = (n, k)$ . Then for any  $j_N(n, k) \in \tilde{X}$ ,

$$j_N \circ S \circ p \circ j_N(n, k) = j_N \circ S \circ h_N \circ p_N(n, k) = j_N \circ S(n) = j_N(n, k),$$

which gives  $j_N \circ S$  is the inverse map to  $p|_{j_N(N \times \{k\})}$ . So  $p|_{j_N(N \times \{k\})}$  is a homeomorphism onto  $N$ .

This gives that the fiber over  $N$  is  $\dot{\cup}_{k \in \mathbb{Z}} j_N(N \times \{k\})$ .

An analogous proof gives that the fiber over  $Y$  is  $\dot{\cup}_{k \in \mathbb{Z}} j_Y(Y \times \{k\})$ .

□

**Lemma 3.18.**  $\text{Aut}(p) = \langle \tau \rangle$ .

*Proof.* By Remark 3.15,  $\tau$  is a homeomorphism on  $\tilde{X}$  and by proposition 3.16,  $p \circ \tau = p$ . Thus  $\tau \in \text{Aut}(p)$  and so  $\langle \tau \rangle \subset \text{Aut}(p)$ .

Let  $\phi \in \text{Aut}(p)$ , then by definition  $p = \phi \circ p$ . Let  $\tilde{y} = j_Y(y, k) \in j_Y(Y \times \mathbb{Z}) \subset \tilde{X}$ . Then  $p(\tilde{y}) \in Y \subset X$ . We know that for all  $x \in Y \subset X$ ,  $p^{-1}(x) = j_Y(x \times \mathbb{Z})$ . So  $p^{-1} \circ p(\tilde{y}) = j_Y(p(y) \times \mathbb{Z})$ . Also since  $p = \phi \circ p$ , then  $\phi(\tilde{y}) \in p^{-1} \circ p(\tilde{y})$  so  $\phi(\tilde{y}) = (p(y), m)$  for some  $m$ . However,  $\tau(j_Y(Y \times \{k\})) = j_Y(Y \times \{k+1\})$ , so  $\tau^{m-k}(\tilde{y}) = (p(y), m) = \phi(\tilde{y})$ . By uniqueness of lifts,  $\phi = \tau^{m-k}$ .

If  $\tilde{y} = j_N(y, k) \in j_N(N \times \mathbb{Z})$ , then a similar argument shows that  $\phi = \tau^{m-k}$ . Since  $\tilde{X} = j_N(N \times \mathbb{Z}) \cup j_Y(Y \times \mathbb{Z})$ , then  $\phi \in \langle \tau \rangle$  giving that  $\text{Aut}(p) \subset \langle \tau \rangle$ . □

**Note 3.19.** By the construction of  $\tilde{X}$ ,  $\tau$  has infinite order, thus  $\text{Aut}(p)$  is infinite cyclic. This defines a  $t$ -action on  $\tilde{X}$  by  $t^k(\tilde{x}) = \tau^k(\tilde{x})$ , by viewing  $\langle t \rangle$  as the infinite cyclic group. This action acts as a shift within  $\tilde{X}$ .

*Remark 3.20.* Now  $p$  is called a *regular covering* because  $\text{Aut}(p) = \langle \tau \rangle$  acts transitively on  $p^{-1}(x)$  for all  $x \in X$ . A consequence of regular coverings gives that  $p_*(\pi_1(\tilde{X})) \trianglelefteq \pi_1(X)$  and  $\pi_1(X)/p_*\pi_1(\tilde{X}) = \text{Aut}(p)$ .

**Theorem 3.21.**  $H_1(\tilde{X})$  is the Alexander module of  $K$ .

*Proof.* Let  $G = \pi_1(X)$ , then by definition the Alexander module  $A_K = G'/G''$ . By Lemma 3.18,  $\text{Aut}(p)$  is infinite cyclic and by Remark 3.20  $G/p_*\pi_1(\tilde{X}) = \text{Aut}(p)$ . Also, by Proposition 2.1,  $G/G' = H_1(X)$  is infinite cyclic. Thus  $G/G' \cong G/p_*\pi_1(\tilde{X})$  which gives  $G' \cong p_*\pi_1(\tilde{X})$ . Since  $\tilde{X}$  is path connected then  $p_*$  is injective which gives that  $\pi_1(\tilde{X}) \cong p_*(\pi_1(\tilde{X})) \cong G'$ . Thus,  $H_1(\tilde{X}) \cong \pi_1(\tilde{X})/(\pi_1(\tilde{X}))' \cong G'/G'' = A_K$ .  $\square$

Recall that the purpose of the construction of the infinite cyclic cover is to understand the relationship between  $\Delta_K$  and the Seifert matrix, with the ultimate goal to prove Theorem 3.11. Theorem 3.21 is the bridge between these two topics. The next series of results gather information about  $H_1(\tilde{X})$  building to a proof of Theorem 3.11.

**Lemma 3.22.** [10] *Let  $M$  be a Seifert surface with  $[a_1], \dots, [a_{2g}]$  is a basis for  $H_1(\mathring{M})$  found in remark 3.6. Then a basis  $[\alpha_1], \dots, [\alpha_{2g}]$  for  $H_1(Y)$  can be found such that  $\text{lk}(\alpha_i, a_j) = \delta_{i,j}$ .*

*Proof.* As described in remark 3.6, each punctured compact orientable manifold with connected boundary and genus  $g$  is homeomorphic to a disk with  $2g$  "handles" attached. The basis elements  $a_i$ 's are loops that run once through each handle. Figure 15 shows the claimed basis elements  $\alpha_i$ 's. Clearly  $\alpha_i$ 's have the property  $\text{lk}(\alpha_i, a_j) = \delta_{i,j}$ .

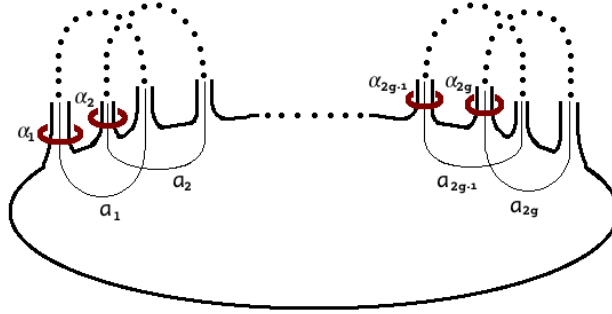


FIGURE 15. choosing  $\alpha$ 's

To show that the  $[\alpha_i]$ 's generate  $H_1(Y)$ , it suffices to show that each handle requires exactly one generating loop element.

First, we will show that each band can have at most one generator. Consider a crossing and the Wirtinger presentation of the surface complement.

Figure 16 gives a relation on the generators  $\alpha\beta^{-1}\alpha^{-1}\gamma = 1$  in  $\pi_1(S^3 - M)$ , but  $H_1(Y) = \pi(S^3 - M)^{ab}$  is abelian, so  $\beta = \gamma$  in  $H_1(Y)$ . This can be interpreted as the generator of the homology can slide under a different handle. Thus each handle requires at most one generator.

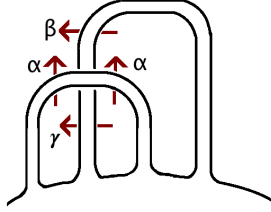


FIGURE 16. Wirtinger presentation for a basic crossing

To show each handle has at least one generator, it suffices to show that there is no relation between the generators. By a Seifert Van Kampen theorem, the Wirtinger presentation is a presentation for the fundamental group of the complement of the knot, and similarly the complement of the Seifert surface. There exists a map from  $\pi_1(Y) \rightarrow \mathbb{Z}^{2g}$  by sending all degenerate generators to the only needed generators. Since  $H_1(Y)$  is the abelianization of  $\pi_1(Y)$  then there is also a homomorphism from  $H_1(Y) \rightarrow \mathbb{Z}^{2g}$ . If there was a linear dependence in  $H_1(Y)$  then there would be a linear dependence in  $\mathbb{Z}^{2g}$ , which there is not. So the  $[\alpha_i]$ 's are linearly independent and actually form a basis for  $H_1(Y)$ .  $\square$

Define  $o_Y : Y \rightarrow \tilde{X}$  as the following composition.

$$Y \hookrightarrow Y \times \{0\} \hookrightarrow Y \times \mathbb{Z} \xrightarrow{j_Y} \tilde{X}$$

This map  $o_Y$  gives a way to view elements of  $Y$  in the zeroth level of  $\tilde{X}$ . Now, for any  $a \in H_1(Y)$  define  $\tilde{a} \in \tilde{X}$  as  $\tilde{a} = o_{Y*}(a)$ .

**Lemma 3.23.**  $H_1(\tilde{X})$  is generated as a  $\Lambda$  module by  $\{\tilde{\alpha}_i, i = 1 \cdots 2g\}$ .

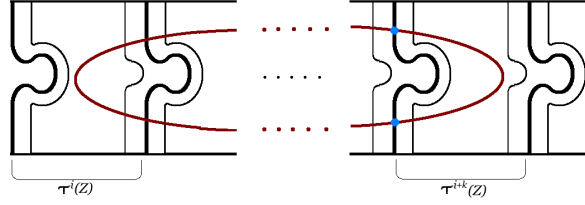
*Proof.* Let  $Z = j_Y(Y \times \{0\}) \cup j_N(N \times \{-1, 0\})$  and so  $\tilde{X} = \cup_{k \in \mathbb{Z}} \tau^k(Z)$ .  $N^+, N^-$  are constructed by thickening  $\mathring{M}$  by the bicollar,  $\mathcal{N}$ . Thus  $\mathcal{N} \circ j_N$  acts as a deformation of  $Z$  into  $j_Y(Y \times \{0\})$ . This gives  $Z$  the homotopy type of  $Y$ . By Lemma 3.22,  $H_1(Y)$  is generated by  $\{\alpha_i\}_{i=1}^{2g}$ . Since  $Z$  and  $Y$  have the same homotopy type, then  $H_1(Z)$  is generated by  $\{\tilde{\alpha}_i\}_{i=1}^{2g}$ .

Let  $C : S^1 \rightarrow \tilde{X}$  be a loop in  $\tilde{X}$ . Now  $\{\tau^k(Z)\}_{k \in \mathbb{Z}}$  is an open cover for  $\tilde{X}$ , so as a compact set,  $C$  is contained in some minimal  $\tau^i(Z)$  and maximal  $\tau^j(Z)$ . Define the  $span(C) = |j - i|$ .

We induct on  $span(C) = k$ .

Base Case:  $k = 0$ . Then  $C \subset \tau^i(Z)$  and so  $\tau^{-i}(C) \subset Z$ .  $H_1(Z)$  generated by  $\{\tilde{\alpha}_i\}_{i=1}^{2g}$ , so  $[\tau^{-i}(C)] = \sum c_i \tilde{\alpha}_i$  for some  $c_i \in \mathbb{Z}$ . Thus



FIGURE 17. Cycle  $C$  in  $H_1(\tilde{X})$ 

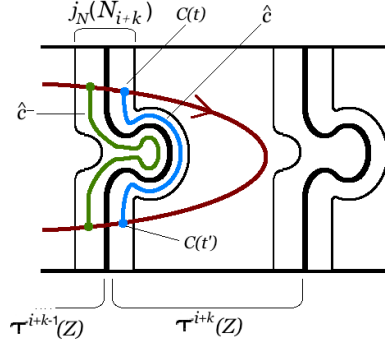
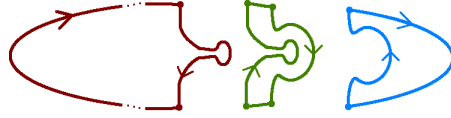
$[C] = \sum [\tau^i(c_i \tilde{\alpha})] = \sum c_i [\tau^i(\tilde{\alpha})] = \sum c_i t^i \tilde{\alpha}$ . Therefore  $[C]$  is a  $\Lambda$  linear combination  $(\tilde{\alpha}_i)_{i=i}^{2g}$ .

**Inductive Step:** Assume that all loops  $\tilde{X}$  with  $\text{span} < k$  are  $\Lambda$  linear combination  $(\tilde{\alpha}_i)_{i=i}^{2g}$ . Let  $C$  be a loop in  $\tilde{X}$  with  $\text{span}(C) = k$ . Then,  $S^1 = \cup_{j=i}^{i+k} C^{-1}(\tau^j(Z))$  and  $\{C^{-1}(\tau^j(Z))\}_{j=i}^{i+k}$  is an open cover for  $S^1$ .  $S^1$  is a compact metric space, so by the Lesbegue Number Lemma, there exists a *finite* subset  $P$  of  $S^1$  such that each component of  $S^1 - P$  is contained one element of the cover. Assign to each partitioned component an index integer  $j$  in  $[i, i+k]$  such that the image of the component is contained in  $C^{-1}(\tau^j(Z))$ . Take a refinement of the partition so that no two consecutive components have the same index. Now, by definition of  $Z$  and the fact that  $C$  is connected, two consecutive components have index that differ by 1. Since  $\tau^j(Z) \cap \tau^{j+1}(Z) = \tau^j(j_N(N \times \{0\})) = j_N(N_j)$ , then the images of the partition points lie in  $j_N(N_j)$  for respective  $j$ .

Let  $\gamma$  be a component with index  $i+k$ , and denote the partition end points of  $\gamma$  by  $t, t' \in S^1$ .  $j_N(N_{i+k})$  is path connected, so let  $\hat{c}$  be a path in  $j_N(N_{i+k})$  from  $C(t')$  to  $C(t)$ . Using the bicollar,  $j_N^{-1}(\hat{c})$  can be slid into  $N^-$ , and so  $\hat{c}$  can be slid to  $\hat{c}^- \subset j_N(N_{i+k-1}^-) \subset \tau^{i+k-1}(Z)$ , as shown in Figure 18.

This "sliding" is a homotopy of  $\hat{c}$  to  $\hat{c}^-$ , so the loop bounded by this homotopy is trivial. Also, the loop  $C(\gamma) \cdot \hat{c}$  lies entirely in  $\tau^{i+k}(Z)$  and is a  $t^{i+k}$  shift of some loop  $c_0$  in  $Z$ . In essence, this process "caps off" a component with index  $i+k$  and shows it is a shift of a loop in  $Z$ . This leaves  $[C]$  as a product of three loops: one which is a shift of a loop in  $Z$ , another which is homotopically trivial, and the last which has one less component in  $\tau^{i+k}(Z)$ . This is schematically shown in figure 19.

By finiteness of the partition, there are finitely many components with index  $i+k$ . Repeat this process for every component with index  $i+k$ . This gives a representation  $[C] = \sum t^{i+k} [c_0^j] + [c']$  where  $c'$  is a loop with  $\text{span}(c') = k-1$  and  $c_0^j$  are loops in  $Z$ . But,  $[c_0^j] = \sum \gamma_m^j \tilde{\alpha}_m \in$

FIGURE 18. Sliding  $\hat{c}$  into  $j_N(N_{i+k}^-)$ FIGURE 19.  $C$  as a product of loops

$H_1(Z)$  for some  $\gamma_m^j \in \mathbb{Z}$ . By the induction hypothesis  $[c']$  is a  $\Lambda$  linear combination of  $(\tilde{\alpha}_i)_{i=i}^{2g}$ . Thus  $C$  is a  $\Lambda$  linear combination of  $(\alpha_i)_{i=i}^{2g}$ .

This finally gives that  $H_1(\tilde{X})$  is  $\Lambda$  generated by  $(\alpha_i)_{i=i}^{2g}$ .  $\square$

Define the maps  $s^\pm : \mathring{M} \rightarrow \mathring{M} \times (-1, 1)$  by  $s^\pm(x) = (x, \pm \frac{1}{2})$ . Note that the bicollar  $\mathcal{N}$  acts as a homotopy between  $s^+$  and  $s^-$ , so the maps  $s^+$  and  $s^-$  are freely homotopic.

Define  $i^+, : \mathring{M} \rightarrow \tilde{X}$  by the following composition:

$$\mathring{M} \xrightarrow{s^+} \mathring{M} \times \{\frac{1}{2}\} \xrightarrow{\mathcal{N}} N^+ \hookrightarrow Y \hookrightarrow Y \times \{0\} \hookrightarrow Y \times \mathbb{Z} \xrightarrow{j_Y} \tilde{X}.$$

Similarly, define  $i^-, : \mathring{M} \rightarrow \tilde{X}$  by the following composition:

$$\mathring{M} \xrightarrow{s^-} \mathring{M} \times \{-\frac{1}{2}\} \xrightarrow{\mathcal{N}} N^- \hookrightarrow N \hookrightarrow N \times \{0\} \hookrightarrow N \times \mathbb{Z} \xrightarrow{j_N} \tilde{X}.$$

So, for any  $a \in H_1(\mathring{M})$  we can define associated  $a^+, a^- \in H_1(\tilde{X})$  by  $a^+ = i^+_*(a)$  and  $a^- = i^-_*(a)$ .

**Lemma 3.24.** *For every  $a \in H_1(\mathring{M})$ ,  $ta^+ = a^-$  in  $H_1(\tilde{X})$ .*

*Proof.* It suffices to show that  $\tau i^+ \simeq i^-$  are freely homotopic as maps from  $\mathring{M}$  to  $\tilde{X}$ , as then  $ta^+ = [\tau i^+(a)] = [i^-(a)] = a^-$  in  $H_1(\tilde{X})$ .

Let  $x \in \overset{\circ}{M}$ , then

$$\begin{aligned}
 \tau i^+(x) &= \tau \circ j_Y \circ \mathcal{N} \circ s^+(x) = \tau \circ j_Y(\mathcal{N}(x, \frac{1}{2}), 0) \\
 &= j_Y(\mathcal{N}(x, \frac{1}{2}), 1) = j_Y \circ b(\mathcal{N}(x, \frac{1}{2}), 0) \\
 &= j_N \circ k_N(\mathcal{N}(x, \frac{1}{2}), 0) = j_N(\mathcal{N}(x, \frac{1}{2}), 0) \\
 &= j_N \circ \mathcal{N} \circ s^+(x)
 \end{aligned}$$

This shows that  $\tau i^+ = j_N \circ \mathcal{N} \circ s^+$ . We also know that  $i^- = j_N \circ \mathcal{N} \circ s^-$ . But since  $s^+ \simeq s^-$ , we get the desired result

$$\tau i^+ = j_N \circ \mathcal{N} \circ s^+ \simeq j_N \circ \mathcal{N} \circ s^- = i^-.$$

□

For the remainder of this section, we use the following notation conventions.

Let  $(a_i)_{i=1}^{2g}$  be generators for  $H_1(\overset{\circ}{M})$  found by the algorithm in Remark 3.6, and  $(\tilde{\alpha}_i)_{i=1}^{2g}$  be generators for  $H_1(\tilde{X})$  corresponding to generators  $(\alpha_i)_{i=1}^{2g}$  for  $H_1(Y)$  found by Lemma 3.22. Define  $\hat{a}_i$  as a loop in  $\overset{\circ}{M}$  such that  $[\hat{a}_i] = a_i \in H_1(\overset{\circ}{M})$ . Define  $\hat{a}_i^\pm = \mathcal{N}(\hat{a}_i, \pm \frac{1}{2})$ .

**Lemma 3.25.**  $H_1(\tilde{X})$  has defining relations  $ta_i^+ = a_i^-$  as a  $\Lambda$  module generated by  $(\tilde{\alpha}_i)_{i=1}^{2g}$ .

*Proof.* Using the notation in the proof of Lemma 3.23,  $\tilde{X} = \cup_{k \in \mathbb{Z}} \tau^k(Z)$  glued along  $\tau^j(Z) \cap \tau^{j+1}(Z) = j_N(N_j)$ . By Lemma 3.24, the Mayer-Vietoris relations for  $\tilde{X}$  as copies of  $Z$  with intersection as copies of  $N$  are  $t^{k+1}a_i^+ = t^k a_i^-$  for all  $k \in \mathbb{Z}$  and  $i \in 1, \dots, 2g$ . But, since  $H_1(\tilde{X})$  is a  $\Lambda$  module, then each of the relators is a unit multiple of the specific relators  $ta_i^+ = a_i^-$  for  $i = 1, \dots, 2g$ .

□

**Proposition 3.26.** For any  $[b] \in H_1(\tilde{Y})$ ,  $[b] = \sum b_j \alpha_j$  where  $b_j = \text{lk}(b, \hat{a}_j)$ .

*Proof.*  $H_1(\tilde{Y})$  is generated by  $(\alpha_i)_{i=1}^{2g}$  by Lemma 3.22, so  $[b] = \sum_j b_j \alpha_j$ . The generators have the property that  $\text{lk}(\alpha_j, \hat{a}_i) = \delta_{ij}$ . By Proposition 3.8,  $\text{lk}$  is bilinear, so

$$\text{lk}(b, \hat{a}_i) = \text{lk}\left(\sum_j b_j \alpha_j, \hat{a}_i\right) = \sum_j b_j \text{lk}(\alpha_j, \hat{a}_i) = \sum_j b_j \delta_{ij} = b_i.$$

□

This completes the background information needed to prove Theorem 3.11.

**Theorem 3.11** *If  $V$  is the Seifert matrix for a knot in  $S^3$ , then  $V^T - tV$  is an Alexander Matrix for the knot.*

*Proof.* Let  $K$  be a knot in  $S^3$ ,  $M$  be a Seifert surface for  $K$  with open bicollar  $\mathcal{N}$  and Seifert form  $f$ . Let  $V$  the Seifert matrix for  $K$  corresponding to a basis  $a_1, \dots, a_{2g}$  for  $H_1(\mathring{M})$  found by the algorithm in Remark 3.6. To show  $V^T - tV$  is an Alexander Matrix, it suffices to show that  $V^T - tV$  is a presentation matrix for  $H_1(\tilde{X})$ , the Alexander module for  $K$ .

Now, Lemma 3.22 gives that  $(\alpha_i)_{i=1}^{2g}$  is a basis for  $H_1(X)$  such that  $\text{lk}(a_i, \alpha_j) = \delta_{i,j}$ . Also, Lemma 3.23 gives that  $H_1(\tilde{X})$  can be presented as a module over  $\Lambda$  with generators  $(\tilde{\alpha}_i)_{i=1}^{2g}$ . Viewing  $\tilde{X}$  as the union  $\tilde{X} = \cup_{k \in \mathbb{Z}} \tau^k(Z)$ , Lemma 3.24 shows that the defining Mayer Vietoris relations are  $a_i^- = ta_i^+$  for  $i = 1, \dots, 2g$ .

So, by definition of  $V$ ,  $V_{ij} = f(a_i, a_j) = \text{lk}(\hat{a}_i, \hat{a}_j^+)$ .

Now by Proposition 3.26,  $[\hat{a}_i^-] = \sum_j \text{lk}(\hat{a}_i^-, \hat{a}_j) \alpha_j$  since  $\hat{a}_i^- \subset Y$ . By definition of  $a_i^- \in H_1(\tilde{X})$ ,

$$\begin{aligned} a_i^- &= i_*^-( [\hat{a}_i^-] ) = [\hat{a}_i^-] = \left[ \sum_j \text{lk}(\hat{a}_i^-, \hat{a}_j) \alpha_j \right] \\ &= \sum \text{lk}(\hat{a}_i^-, \hat{a}_j) [\alpha_j] = \sum \text{lk}(\hat{a}_i^-, \hat{a}_j) \tilde{\alpha}_j. \end{aligned}$$

But, since  $\hat{a}_i^-$  is homotopic to  $\hat{a}_i$  and  $\hat{a}_j$  is homotopic to  $\hat{a}_j^+$  with disjoint homotopies, then by Proposition 3.8,  $\text{lk}(\hat{a}_i^-, \hat{a}_j) = \text{lk}(\hat{a}_i, \hat{a}_j^+) = v_{i,j}$ . So

$$a_i^- = \sum V_{ij} \tilde{\alpha}_j$$

Similarly,

$$a_i^+ = \sum \text{lk}(\hat{a}_i^+, \hat{a}_j) \tilde{\alpha}_j = \sum \text{lk}(\hat{a}_j, \hat{a}_i^+) \tilde{\alpha}_j = \sum V_{ji} \tilde{\alpha}_j.$$

Now imposing that  $a_i^- = ta_i^+$  we get the following:

$$\begin{aligned} \sum_j v_{i,j} \alpha_j &= t \sum_j v_{j,i} \alpha_j \\ \sum_j [v_{i,j} - tv_{j,i}] \alpha_j &= 0 \end{aligned}$$

This interpretation of the Mayer-Vietoris relations shows that  $V - tV^T$  is a presentation matrix for  $A_k$ . Now, by interchanging the  $+$  and  $-$  sides of the bicollar on  $M$ , the new Seifert matrix is the transpose of the old one. This gives that  $V^T - tV$  is a presentation matrix for  $A_k$ .

This proof is due to Rolfsen [10].

□

**Note 3.27.**  $V^T - tV$  is a square matrix presentation matrix for  $A_k$ . Thus there is a well defined generator of the order ideal of  $A_k$ , namely the Alexander polynomial.

To calculate the Alexander polynomial using the Seifert matrix is very difficult, in the sense that one needs to find generators of the homology group of the Seifert surface. While there is an algorithm to do this described in Remark 3.6, this assumes that you know how to view the surface as a disk with handles, which is indeed difficult. However, this description of the Alexander polynomial is very useful for proving properties of the polynomial. In the next section, the Alexander polynomial is proved to satisfy the Skein relations using only the Seifert matrix formulation. In addition, the following two beautiful results have simple proofs using the Seifert matrix.

**Corollary 3.28.**  $\Delta(t) = \pm t^k \Delta(t^{-1})$  for some  $k$ .

*Proof.*

$$\begin{aligned} (V^T - tV)^T &= V - tV^T = t(V^T - t^{-1}V) \\ \det((V^T - tV)^T) &= \det(t\mathbb{I}(V^T - t^{-1}V)) \\ \det(V^T - tV) &= \pm t^k \det(V^T - t^{-1}V) \\ \Delta(t) &= \pm t^k \Delta(t^{-1}). \end{aligned}$$

□

**Corollary 3.29.** Define the degree of a Laurent polynomial to be the difference of the highest and lowest exponents with nonzero coefficients and  $g$  to be the genus of the knot. Then,  $\text{degree}(\Delta(t)) \leq 2g$ .

*Proof.* For a given knot and Seifert surface with genus  $g$ , the Seifert matrix is a  $2g$  by  $2g$  matrix with integer entries. Each entry of  $V - tV^T$  is a linear term, so  $\text{degree}(\Delta(t)) = \text{degree}(\det(V - tV^T)) \leq 2g$ . □

For example, the  $5_1$  and  $7_1$  knots are shown in Figure 20. The Alexander polynomials of these knots were found in Rolfsen's knot index [10].

$\Delta_{5_1}(t) = t^2 + t^{-2} - t - t^{-1} + 1$  which has degree 4. So the genus of  $5_1$  is at least 2.

$\Delta_{7_1}(t) = t^3 + t^{-3} - t^2 - t^{-2} + t + t^{-1} - 1$  which has degree 6. So the genus of  $7_1$  is at least 3.

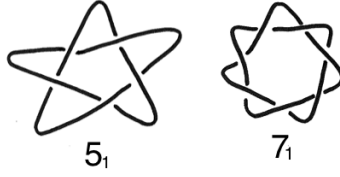


FIGURE 20

The next results give a slightly alternate form of the Seifert matrix description of the Alexander polynomial in preparation for the Conway and the Skein relation section.

**Proposition 3.30.** *If  $R$  is a ring, then for any  $M \in M_{n \times n}(R)$ ,*

$$\pm t^n \det(tM - t^{-1}M^T) = \det(M^T - t^2M).$$

*Proof.* Let  $R$  be a ring and  $M \in M_{n \times n}(R)$ . Let  $\mathbb{I}$  be the identity matrix in  $M_{n \times n}(R)$ . Then

$$t\mathbb{I}(tM - t^{-1}M^T) = -(M - t^2M^T).$$

Taking determinants gives

$$\begin{aligned} \pm t^n \det(tM - t^{-1}M^T) &= \pm \det(t\mathbb{I}) \det(tM - t^{-1}M^T) \\ &= \pm \det(t\mathbb{I}(tM - t^{-1}M^T)) = \pm \det(M - t^2M^T) \end{aligned}$$

as desired. □

**Corollary 3.31.** *If  $M$  is a Seifert matrix for a knot  $K$ , then  $\det(t^{\frac{1}{2}}M - t^{-\frac{1}{2}}M^T)$  is the Alexander polynomial for  $K$ .*

*Proof.* Proposition 3.30 states that

$$\det(M^T - t^2M) = \pm t^n \det(tM - t^{-1}M^T).$$

But  $\Delta_K(t^2) = \det(M - t^2M^T)$  has only even powers of  $t$  in  $\Lambda$ . So

$$\Delta_K(t) = \det(M^T - tM) = \pm t^k \det(t^{\frac{1}{2}}M - t^{-\frac{1}{2}}M^T)$$

where  $\pm t^k \det(t^{\frac{1}{2}}M - t^{-\frac{1}{2}}M^T)$  is actually a polynomial in  $\Lambda$ .

Now  $\pm t^n$  is an invertible element in  $\Lambda$ . Thus  $\Delta(K)$  and  $\det(t^{\frac{1}{2}}M - t^{-\frac{1}{2}}M^T)$  generate the same ideal, namely the Alexander Ideal, in  $\Lambda$ . □

*Remark 3.32.*  $\det(tM - t^{-1}M^T)$  can be thought of as  $\Delta_K(t^2)$ , or  $\Delta_K$  with a change of variable  $t$  to  $t^2$ . Thus establishing a result for  $\det(tM - t^{-1}M^T)$  in turn establishes a result for  $\Delta_K$  with a change of variable. This relationship is very useful in the following sections.

## 4. CONWAY AND THE SKEIN RELATION

The Seifert surface description and the original definition of the Alexander polynomial offer interesting insight, but are generally difficult routes to compute the Alexander polynomial. The result of Conway and the Skein relation is the most useful description for computation of the Alexander polynomial. The main result of this section is theorem 4.3.

**Definition 4.1.** Let  $L$  be a knot or link. Define  $L_+$ ,  $L_-$  and  $L_0$  by isolating and changing one crossing of  $L$  as shown in figure 21.

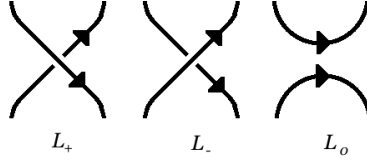


FIGURE 21

**Definition 4.2.** A function  $f$  satisfies the **Skein Relation** if for any knot or link  $L$

$$f(L_+) - f(L_-) = (t^{-\frac{1}{2}} - t^{\frac{1}{2}})f(L_0).$$

**Theorem 4.3.** Any knot invariant satisfying the Skein relation and has the value 1 on the trivial knot is the Alexander polynomial.

The structure of the proof is first to show that the Alexander polynomial satisfies these properties. Then show that these properties completely define a knot invariant. However, the Skein relations heavily rely on the use of links. Up until now, all of the results in this paper have been for knots, and are not necessarily true for links. Since the focus of this paper is knots, the following results about links will simply be stated and not proved.

**Definition 4.4.** A **splittable link** is a link that can be separated by a 2-sphere embedded in  $S^3$ .

**Lemma 4.5.** If  $L$  is a splittable with at least two components, then  $\Delta_L = 0$ .

A proof of this result can be found in [3] and [1], Corollary 9.17.

**Theorem 4.6.** The Alexander polynomial satisfies the Skein relation  $\Delta(L_+) - \Delta(L_-) = (t^{-\frac{1}{2}} - t^{\frac{1}{2}})\Delta(L_0)$ . ([3] pg 162)

*Proof.* Let  $M_+$ ,  $M_-$  and  $M_0$  be Seifert surfaces constructed using Seifert's algorithm for  $L_+$ ,  $L_-$  and  $L_0$  respectively, and  $\mathcal{N}_\pm$  be the open bicollars on  $M_\pm$ .

Consider cases on  $L_+$ ,  $L_-$  and  $L_0$ .

- Case 1: Suppose  $L_+$  is a split link. (The case if  $L_-$  is a split link is analogous.)

If  $L_+$  is a split link, then both  $L_-$  and  $L_0$  are also split links. By Lemma 4.5,  $\Delta(L_+) = \Delta(L_-) = \Delta(L_0) = 0$ , so trivially  $\Delta(L_+) - \Delta(L_-) = (t^{-\frac{1}{2}} - t^{\frac{1}{2}})\Delta(L_0)$ .

- Case 2: Suppose  $L_0$  is a split link, but  $L_+$  and  $L_-$  are not split links.

Then by Lemma 4.5,  $\Delta(L_0) = 0$ . Now  $L_+$ ,  $L_-$  and  $L_0$  are of the form show in the figure.

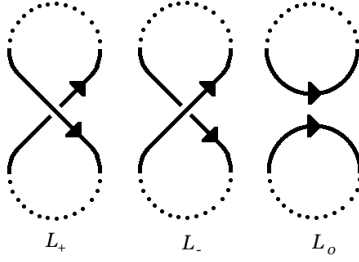


FIGURE 22. crossings relations for  $L$

Now,  $L_+$  is equivalent to  $L_-$  by a  $2\pi$ -twist, (this is proved using grid diagrams in Proposition 5.5). Since  $\Delta$  is a knot invariant, then  $\Delta(L_+) = \Delta(L_-)$  and so

$$\Delta(L_+) - \Delta(L_-) = 0 = \Delta(L_0) = (t^{-\frac{1}{2}} - t^{\frac{1}{2}})\Delta(L_0).$$

- Case 3: Suppose  $L_+$ ,  $L_-$  and  $L_0$  are all NOT split links.

Since  $M_0$  and  $M_-$  were constructed using Seifert's algorithm, then  $M_0$  and  $M_-$  differ only locally by a half twist, as shown in figure 23.

Let  $\{[a_1], \dots, [a_n]\}$  be a basis for  $H_1(M_0)$  found by the algorithm described in Remark 3.6. Now,  $a_i \subset M_-$  as each  $a_i$  lies in a part of the surface that is unchanged by the local crossing. Let  $b$  be the cycle in  $M_-$  that passes once through the added half twisted band as shown in figure 23.

Claim:  $\{[a_1], \dots, [a_n], [b]\}$  is a basis for  $H_1(M_-)$ .

Now, up to homotopy type,  $M_- = M_0 \vee S^1$ . The Mayer Vietoris Sequence of wedge products yields that  $H_1(M_-) \cong$



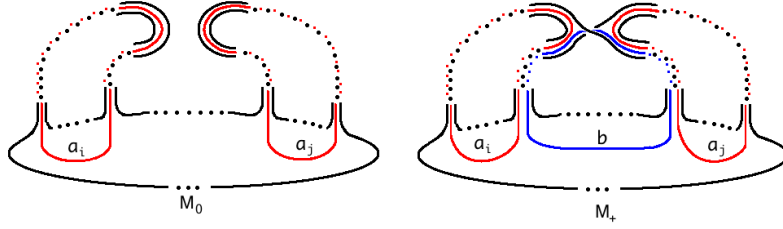


FIGURE 23

$H_1(M_0) \oplus H_1(S^1)$ . Since  $b$  generates  $S^1$  then  $\{[a_1], \dots, [a_n], [b]\}$  is a basis for  $H_1(M_-)$ .

Next, for any loop  $a$  in  $M_\pm$ , denote  $a^+ = \mathcal{N}_\pm(a, \frac{1}{2})$ . Let  $\alpha_i = lk(b, a_i^+)$ ,  $\gamma_i = lk(a_i, b^+)$  and  $\beta = lk(b, b^+)$ . From this, the Seifert matrix for  $M_-$  is given below, where  $\mathbb{M}_0$  is the Seifert matrix for  $M_0$ .

$$\mathbb{M}_- = \begin{bmatrix} & & \gamma_1 \\ & \mathbb{M}_0 & \vdots \\ & & \gamma_n \\ \alpha_1 & \cdots & \alpha_n & \beta \end{bmatrix}$$

Analogously, the Seifert surface for  $M_+$  will be  $M_0$  with an extra twist in the opposite direction. The same argument gives that  $\{a_1, \dots, a_n, c\}$  is a basis for  $H_1(M_+)$ , where  $c$  is the cycle that passes once through the added half twist analogously to  $b$  in  $M_-$ .

Claim:  $lk(c, c^+) = \beta - 1$

By construction,  $b$  and  $c$  are basically the same cycle except  $b$  passes through a negatively oriented half twist and  $c$  passes through a positively oriented half twist. So, the linking numbers  $lk(b, b^+)$  and  $lk(c, c^+)$  can only differ at the local crossing in the half twist, figure 24.

Consider only the crossing induced in the half twist. Now  $b$  crosses over  $b^+$ , so no number is assigned to the over crossing. However,  $c$  crosses under  $c^+$ , adding  $-1$  to the  $lk(c, c^+)$ . Thus  $lk(c^+, c) = lk(b^+, b) - 1 = \beta - 1$ . This proves the claim.

Next, let  $\nabla(L) = \det(tM - t^{-1}M^T)$ . By the change of variable  $t$  to  $t^2$ , to show that  $\Delta$  satisfies  $\Delta(L_+) - \Delta(L_-) = (t^{-\frac{1}{2}} - t^{\frac{1}{2}})\Delta(L_0)$ , it suffices to show that  $\nabla$  satisfies the relation  $\nabla(L_+) - \nabla(L_-) = (t^{-1} - t)\nabla(L_0)$ , by Remark 3.32. So,

$$\nabla(L_+) = \det(t\mathbb{M}_+ - t^{-1}\mathbb{M}_+^T) =$$

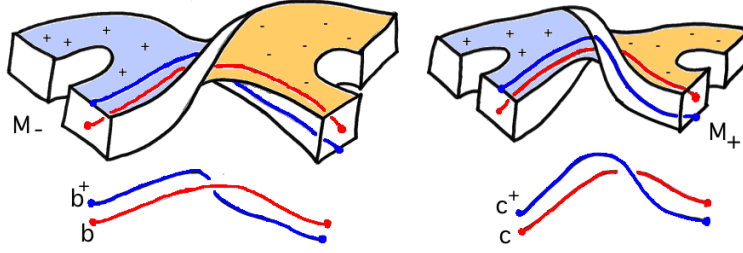


FIGURE 24

$$\begin{aligned}
&= \det \begin{bmatrix} & & & t\gamma_1 - t^{-1}\alpha_1 \\ & & & \vdots \\ & t\mathbb{M}_0 - t^{-1}\mathbb{M}_0 & & t\gamma_n - t^{-1}\alpha_n \\ t\alpha_1 - t^{-1}\gamma_1 & \cdots & t\alpha_n - t^{-1}\gamma_n & t(\beta - 1) - t^{-1}(\beta - 1) \end{bmatrix} \\
&= (t(\beta - 1) - t^{-1}(\beta - 1)) \det(t\mathbb{M}_0 - t^{-1}\mathbb{M}_0) + f(t^2, \alpha_i, \gamma_j) \\
&= (t(\beta - 1) - t^{-1}(\beta - 1)) \nabla(L_0) + f(t^2, \alpha_i, \gamma_j).
\end{aligned}$$

Where  $f$  is a polynomial function of  $t^2, \alpha_i$  and  $\gamma_j$  for  $i, j \in \{1, \dots, n\}$ . Similarly, with the same function  $f$ ,

$$\nabla(L_-) = \det(t\mathbb{M}_- - t^{-1}\mathbb{M}_-^T) = (t\beta - t^{-1}\beta) \nabla(L_0) + f(t^2, \alpha_i, \gamma_j).$$

This gives the relation

$$\begin{aligned}
\nabla(L_+) - \nabla(L_-) &= (t(\beta - 1) - t^{-1}(\beta - 1)) \nabla(L_0) - (t\beta - t^{-1}\beta) \nabla(L_0) \\
&= (t\beta - t + t^{-1}\beta - t^{-1} - t\beta + t^{-1}\beta) \nabla(L_0) \\
&= (t^{-1} - t) \nabla(L_0).
\end{aligned}$$

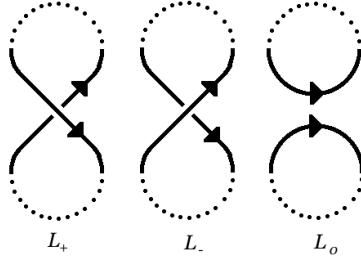
□

Next, we will show that a knot invariant satisfying the Skein relation can be computed using only its value on the unknot. Thus, a knot invariant is completely determined by the Skein relation and value on the unknot. The following is based on Kauffman's approach to Skein theory in [6].

Let  $\nabla$  be a knot invariant satisfying the Skein relation and has value 1 on the trivial knot.

**Lemma 4.7.** *If  $L$  is a split link with at least two components, then  $\nabla_L = 0$ . ([6] pg 20).*

*Proof.* If  $L$  has 2 components, then up to isotopy  $L$  is as shown in Figure 25.  $\nabla_L$  must satisfy the Skein relation, so  $\nabla_{L_+} - \nabla_{L_-} = (t^{-\frac{1}{2}} - t^{\frac{1}{2}}) \nabla_{L_0}$ , for  $L_+$  and  $L_-$  shown in Figure 25.

FIGURE 25. crossings relations for  $L$ 

Now,  $L_+$  is equivalent to  $L_-$  by a  $2\pi$ -twist, (this is proved using grid diagrams in proposition 33). Since  $\nabla$  is a knot invariant, then  $\nabla_{L_+} = \nabla_{L_-}$  and so  $\nabla_{L_0} = \nabla_{L_+} - \nabla_{L_-} = 0$ .

Inductively, it follows that if  $L$  is a split link with  $n$  components, then  $\nabla_L = 0$ , for all  $n \in \mathbb{N}$ .

□

**Definition 4.8.** A *resolving tree* for a knot  $K$  is a binary tree diagram with  $K$  at the root of the tree. At each stage of the tree one crossing is isolated and changed according to the Skein relation so that every triplet (parent, left child, right child) is of the form  $(K_+, K_0, K_-)$  or  $(K_-, K_0, K_+)$ . Each node diagram may be replaced by an isotopic diagram before applying the Skien relation. [3] pg 167

Now the following describes a recursive algorithm to calculate  $\nabla$  for knot  $K$ :

First create a resolving tree diagram for  $K$ . The terminal nodes will be spilt links. By assumption,  $\nabla$  is 1 on the unknot and  $\nabla$  is 0 on split links by Lemma 4.7. Using the Skien relation,  $\nabla_K$  is a linear combination of  $\nabla$  applied to each terminal node. For ease of computation, index the nodes of the tree.

An example calculation for the figure 8 knot from a resolving tree in Figure 26 follows.

Now by assumption,  $\nabla K_{00} = \nabla K_- = 1$  as  $K_{00}$  and  $K_-$  are the trivial knot. By Lemma 4.7,  $\nabla K_{0-} = 0$  as  $K_{0-}$  is a split link. So, applying the Skein relation to the bottom tier, we get:

$$\begin{aligned} \nabla K_{0+} - \nabla K_0 &= (t^{-\frac{1}{2}} - t^{\frac{1}{2}}) \nabla K_{00} \\ \Rightarrow \nabla K_0 &= -(t^{-\frac{1}{2}} - t^{\frac{1}{2}}). \end{aligned}$$

Then apply the Skein relation to the second tier to get:

$$\nabla K_+ - \nabla K_- = (t^{-\frac{1}{2}} - t^{\frac{1}{2}}) \nabla K_0$$

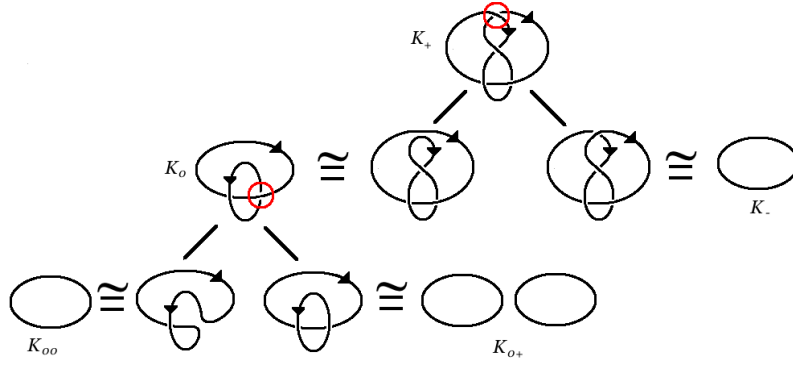


FIGURE 26. Resolution tree for the figure 8 knot

$$\begin{aligned} \Rightarrow \nabla_{K_+} - 1 &= -(t^{-\frac{1}{2}} - t^{\frac{1}{2}})(t^{-\frac{1}{2}} - t^{\frac{1}{2}}) \\ &\Rightarrow \nabla_{K_+} = 3 + t + t^{-1}. \end{aligned}$$

Thus giving that  $\nabla$  of the figure eight knot is  $3 + t + t^{-1}$ .

## 5. MINESWEEPER MATRIX

The Minesweeper Matrix is another description of the Alexander polynomial that is easy to compute. Grid diagrams are a representation of knots and links that are used to describe the Minesweeper Matrix. This section develops some basic theory of grid diagrams and then describes the Minesweeper Matrix.

**Definition 5.1.** A *grid diagram* is a two dimensional square grid such that each square within the grid is decorated with an  $x$ ,  $o$  or is left blank. This is done in a manner such that every column and every row has exactly one  $x$  and one  $o$  decoration.

**Definition 5.2.** The *grid number* of a grid diagram is the number of columns (or rows) in the grid.

See figure 27 for an example. This section follows the grid notation by Manolescu, Ozsváth, Szabó and Thurston in [8] (see also [7]) with the convention that the rows and columns are numbered top to bottom and left to right, respectively.

A grid diagram is associated with a knot by connecting the  $x$  and  $o$  decorations in each column and row by a straight line with the convention that *vertical lines cross over horizontal lines*. These lines form strands of the knot, and removing the grid leaves a projection of the knot. As a result, grid diagrams represent particular planar projections of knots. This process is illustrated in figure 28. The *knot type of a grid* is the knot type of the knot associated with the grid.

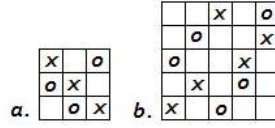


FIGURE 27. Example grid diagrams with grid numbers 3 and 5.

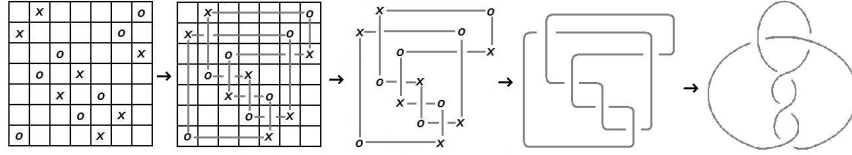


FIGURE 28

There are three grid moves used to relate grid diagrams: commutation, cyclic permutation and stabilization. These play a role analogous to the Reidemeister moves for knot diagrams [9]. Following the notation from [8], the three grid moves are as follows:

- (1) Commutation interchanges two consecutive rows or columns of a grid diagram. This move preserves the grid number as shown in figure 29. Even though commutation may be defined for any two rows or columns, it is only permitted if the commutation preserves the knot type of the grid. Throughout this section, all discussed commutations preserve the knot type.

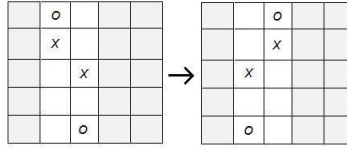


FIGURE 29. An example of column commutation

- (2) Cyclic permutation preserves the grid number and removes an outer row/column and replaces it to the opposite side of the grid. See figure 30.
- (3) Stabilization, also known as kink addition or removal, does not preserve the grid number. A kink may be added to the right or left of a column, and above or below a row. To add a kink to column  $c$ , insert an empty row between the  $x$  and  $o$  markers of the column  $c$ . Then insert an empty column to the right or left of column  $c$ . Move either the  $x$  or  $o$  decoration in column  $c$  into

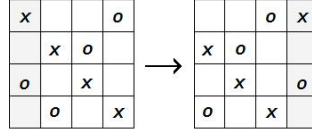


FIGURE 30. An example of column permutation

the adjacent grid square in the added column. Complete the added row and column with  $x$  and  $o$  decorations appropriately. See figure 31. To add a kink to a row, switch the notions of column and row. To remove a kink, follow these instructions in reverse order. As shown, adding a kink increases the grid number by 1 while removing a kink reduces the grid number by 1.

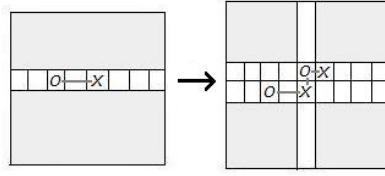


FIGURE 31. An example of stabilization, kink addition

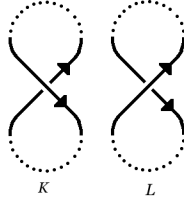
The following theorem, due to Cromwell [2] and Dynnikov [4], explicates the relationship between grid diagrams, knots and the three grid moves.

**Theorem 5.3.** (*Cromwell [2], Dynnikov [4]*) *Let  $G_1$  be a grid diagram representing knot  $K_1$  and  $G_2$  be a grid diagram representing knot  $K_2$ .  $K_1$  and  $K_2$  are equivalent knots if and only if there exists a sequence of commutation, stabilization and cyclic permutation grid moves to relate  $G_1$  to  $G_2$ .*

In other words, the three grid moves form an equivalence relation on the set of grid diagrams, and two grid diagrams are equivalent if and only if they represent the same knot. The three grid moves play a role similar to the Reidemeister moves for knot diagrams [9].

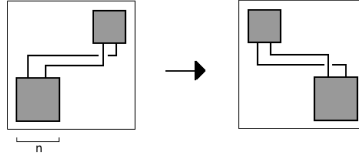
As a slight aside, theorem 4.6 required a property of knots that are related by a  $2\pi$  twist. This property is described and proved below.

**Definition 5.4.** Two knots  $K$  and  $L$  are related by a  $2\pi$  twist if  $K$  and  $L$  differ in exactly one crossing as show in figure 32.

FIGURE 32.  $2\pi$  twist

**Proposition 5.5.** *If two knots  $K$  and  $L$  differ by a  $2\pi$ -twist then  $K = L$  as knots.*

*Proof.* Consider the grid diagrams associated to  $K$  and  $L$ ,  $D_K$  and  $D_L$  respectively.  $D_K$  has a block of width  $n$  connected to another block by two strands crossing once.  $n$  applications of cyclic permutation of the first column relates  $D_K$  to  $D_L$ .

FIGURE 33. cyclic permutation  $n$  times

□

**Theorem 5.6.** [11] *There exists a sequence of commutation and stabilization grid moves to accomplish any cyclic permutation.*

*Proof.* Figure 34 shows a sequence of stabilization and commutation grid moves that accomplished a cyclic permutation move. □

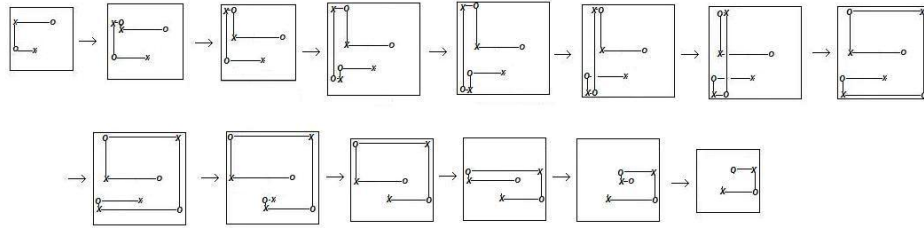


FIGURE 34

**Corollary 5.7.** *Two knot diagrams represent the same knot type if and only if there exists a sequence of commutation and stabilization grid moves to relate the diagrams.*

### 5.1. Minesweeper Matrix.

Fix an orientation and grid diagram with grid number  $n$  for the knot. Leave the grid behind the knot. Follow Seifert's algorithm to create the Seifert circles. At each interior intersection for the grid lines place a  $w_c$ , where  $w_c$  is the *wind* of Seifert circle  $c$  that contains that grid intersection. Calculate the wind of the circle in the following manner:

- Let  $w_c^p$  be the partial wind for Seifert circle  $c$ .
- $w_c^p = 1$  for a circle with a clockwise orientation.
- $w_c^p = -1$  for a circle with a counter-clockwise orientation.
- If circle  $c$  is not contained in another circle, let  $w_c = w_c^p$ .
- If circle  $c$  is contained in another circle (or circles)  $H$ , let  $w_c = w_c^p + w_H$ .

If a grid intersection lies outside the closed curves of the knot and Seifert circles, then the wind of this intersection is 0 and  $t^{w_c} = 1$ .

Remove the grid and the knot leaving behind only the added  $w_c$ 's in organized columns and rows. Create a matrix with entries  $t^{w_c}$  and add a row of 1's and a columns of 1's, see Figure 35. Hence there is a matrix of dimension  $n$  by  $n$ . This matrix is called the **Minesweeper Matrix**, denoted  $\mathcal{M}$ .

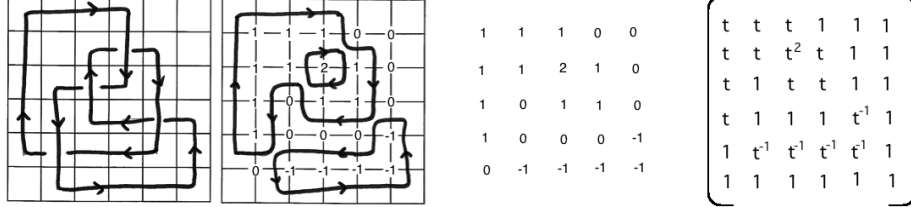


FIGURE 35. Minesweeper matrix for the figure 8 knot

**Theorem 5.8.** [8] *The Alexander polynomial for a knot  $k$  is*

$$\Delta_k(t) = (1 - t)^{-(n-1)} \det(\mathcal{M})$$

*where  $n$  is the grid number and  $\mathcal{M}$  is the Minesweeper matrix.*

To prove this theorem, it suffices to show that  $(1 - t)^{-(n-1)}\mathcal{M}$  is a knot invariant and satisfying the Skein relations. This will be proven in separate pieces.

**Note 5.9.** The difference between any two neighboring  $t^{w_c}$ 's can be only  $\pm t^{\pm i}(1 - t)$  or 0. If the two  $t^{w_c}$ 's are in the same Seifert circle or both lie outside of a closed curve then their winds are the same and the difference is 0. If the two  $t^{w_c}$ 's are in different Seifert circle then the inner Seifert circle's wind,  $w_{c_i}$ , can only be  $\pm 1$  more than the wind of outer Seifert circle,  $w_{c_o}$ . Hence,



$$t^{w_{co}} - t^{w_{ci}} = t^{w_{co}} - t^{(w_{co} \pm 1)} = \pm t^{w_{co}} (1 - t).$$

If one  $t^{w_c}$  lies outside of a closed curve then it has a wind,  $w_r$ , of 1. If the other lies inside a Seifert circle then it has a wind,  $w_s$ , of  $\pm 1$ . If  $w_s = 1$  then  $t^{w_s} - t^{w_r} = t - 1 = -(1 - t)$ . If  $w_s = -1$  then  $t^{w_s} - t^{w_r} = t^{-1} - 1 = t^{-1}(1 - t)$ .

**Theorem 5.10.** *For Minesweeper Matrix  $\mathcal{M}$ ,  $(1 - t)^{-(n-1)} \det(\mathcal{M})$  is a knot invariant.*

*Proof.* By the Corollary 5.7, to prove  $(1 - t)^{-(n-1)} \mathcal{M}$  is a knot invariant, it is enough to show that the  $\mathcal{M}$  is invariant under the grid moves commutation and stabilization.

- Commutation

Commuting two grid rows or columns corresponds to commuting two rows or columns in the matrix  $M$ . Commuting two rows or columns in a matrix multiplies the determinant by  $-1$ , an invertible element in  $\Lambda$ .

- Stabilization or destabilization

Adding or removing a kink does not change the Seifert circles, but rather changes their size or shape. So intuitively it is clear that this would not change the Alexander polynomial.

Let  $\mathcal{M}$  be a minesweeper matrix for a grid diagram for a knot. When adding a kink, a new row and column are added to the grid and contribute a new row and column to the matrix. Call this new matrix  $\mathcal{M}_s$ . Within the **grid**, the new row and column are completely different from the neighboring rows and columns. Within the knot and  $\mathcal{M}_s$ , the new row and column only stretch the original Seifert circles and ultimately contribute an almost identical row and column from  $\mathcal{M}$  to  $\mathcal{M}_s$ . The new row and column will differ from an original row and column from  $M$  in exactly one entry where the kink was added, hence *almost* identical, see figure 36

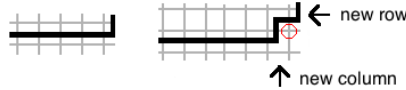


FIGURE 36

Depending on the direction of the kink and the relative position of the original  $x$  and  $o$ , the new row and column will be almost identical to different rows and columns from  $\mathcal{M}$ . The reason for this almost identity is when the new row and column

are added, all other rows and columns are extended through the new row and column stretching the strands of the knot through the new row and column in exactly same way as the neighboring rows and columns. So the grid square with the kink is the only different square.

Adding a multiple of a row to another row or a multiple of a column to another column does not change the determinant. So, for each direction of kink, subtract the almost identical column from the new column. Every entry in the new column will cancel to zero except the entry where the kink was added. Since the two columns are consecutive, the entry with the kink will be  $\pm t^{\pm i}(1 - t)$ , as proved in the Note 5.9. Similarly, subtract the almost identical row from the new row. The same cancellation will happen in every entry except the entry with the kink. Since the new column was zeroed out, the almost identical row has a zero in the entry of the intersection of the new column and the almost identical row. Hence only a zero was added to the kink entry leaving it unchanged. So  $\mathcal{M}_s$  has a row and column of zeros except the intersection entry of the zeroed row and column is  $\pm t^{\pm i}(1 - t)$ .

To calculate the determinant of  $\mathcal{M}_s$ , use the expansion by cofactors method. Expand by either the new row or column. Since all but one entry is zero, the determinant of  $\mathcal{M}_s$  will be  $\pm t^{\pm i}(1 - t)$  times the determinant of the minor matrix created by deleting the new row and column which is precisely the determinant of  $\mathcal{M}$ .

So,  $\det(\mathcal{M}_s) = \pm t^{\pm i}(1 - t)\det(\mathcal{M})$ .

And thus,

$$\begin{aligned}\Delta_k(t) &= \pm t^j(1 - t)^{-((n+1)-1)}\det(\mathcal{M}_s) \\ &= \pm t^j(1 - t)^{-((n+1)-1)}(\pm t^{\pm i})(1 - t)\det(\mathcal{M}) \\ &= \pm t^{(j \pm i)}(1 - t)^{-(n-1)}\det(\mathcal{M}).\end{aligned}$$

□

**Theorem 5.11.**  $(1 - t)^{-(n-1)}\det(\mathcal{M})$  satisfies the Skein relation and has value 1 on the trivial knot.

*Proof.* Firstly, figure 37 shows a grid diagram and minesweeper matrix for the trivial knot. Thus

$$(t - 1)^{n-1}\det(\mathcal{M}) = (t - 1)^{-1}(t - 1) = 1.$$

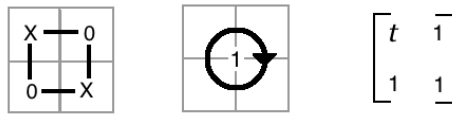


FIGURE 37. Grid diagram, Seifert circle and Minesweeper matrix for the trivial knot

Next, the Minesweeper matrix satisfies the Skein relation. To prove this, consider how the grid diagram is changed under the crossing resolutions, in figure 38, and compare minesweeper matrices for each grid.

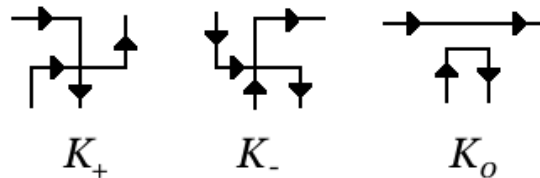


FIGURE 38. Local segments of grid diagram corresponding to different crossing resolutions

Note that these local crossing resolutions only change the minesweeper matrices in one row. Expanding the determinant along this row yields comparable representations of the Minesweeper matrices that satisfy the Skein relation. This tedious computation is very straight forward and is presented nicely in [12] pg 13.

□

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