Curves in Non-Orientable Surfaces

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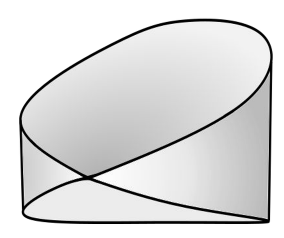
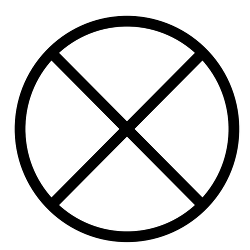
What are Non-Orientable Surfaces?

The **Möbius band** is a non-orientable surface with only one side and one boundary curve -- a circle.



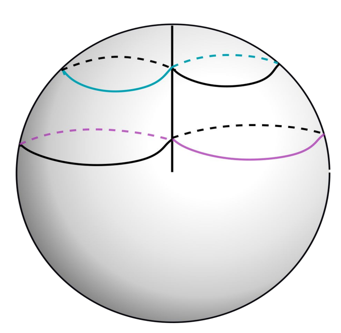
The **cross-cap** is a reconfiguration of the Möbius band where the boundary circle looks like a true circle.



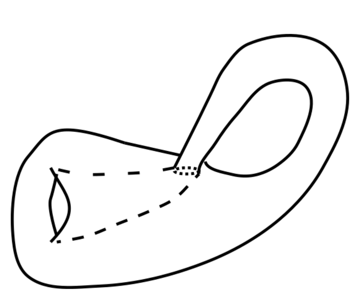
Cross caps are in 4D, not 3D!

Scan for helpful animation!

Gluing the boundary of a disc to the boundary of a cross-cap yields the **real projective plane.**

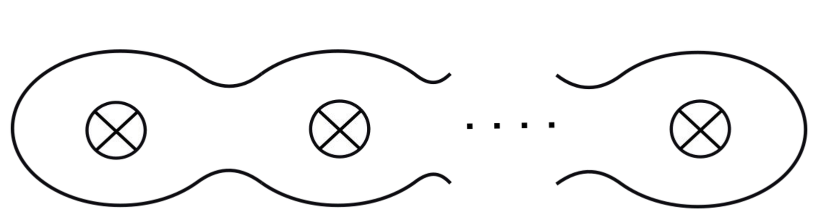


Gluing two cross-caps together along their boundary yields the **Klein bottle.**



Classification Theorem

All non-orientable surfaces are formed by glueing some number of cross-caps to a sphere.



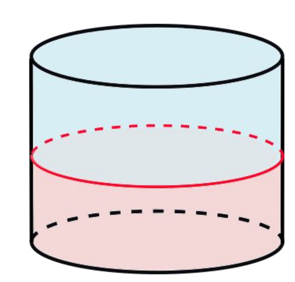
N_g

Our Project

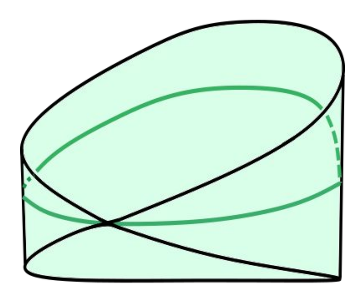
Generalize known results about collections of curves in orientable surfaces to the non-orientable case.

Simple Closed Curves in Surfaces

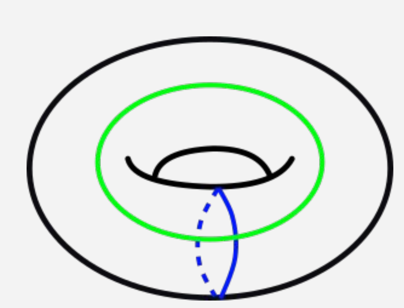
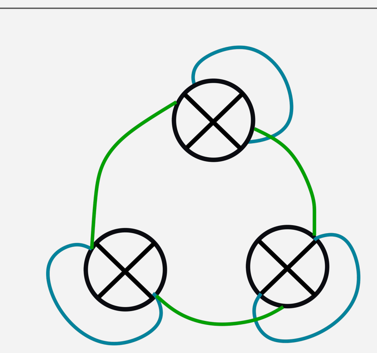
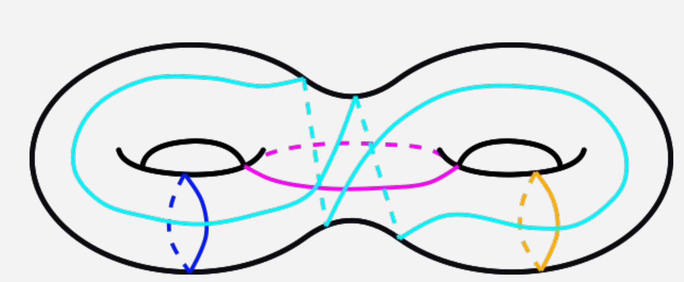
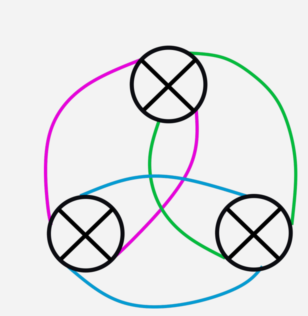
For this project, curves are homotopy classes of closed loops with no self intersections.



2-sided curves are the core of a cylinder.

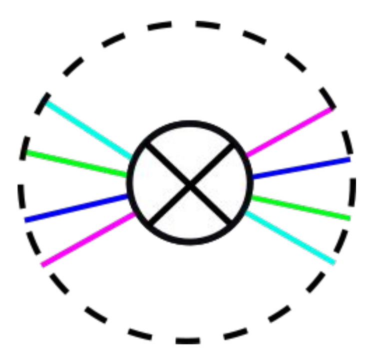


1-sided curves are the core of a Möbius band.

Orientable	vs. Non-Orientable
All curves are 2-sided.	Curves can be either 1-sided or 2-sided.
 curves in the torus	 1-sided curves in N_3
 curves in the genus 2 surface	 2-sided curves in N_3

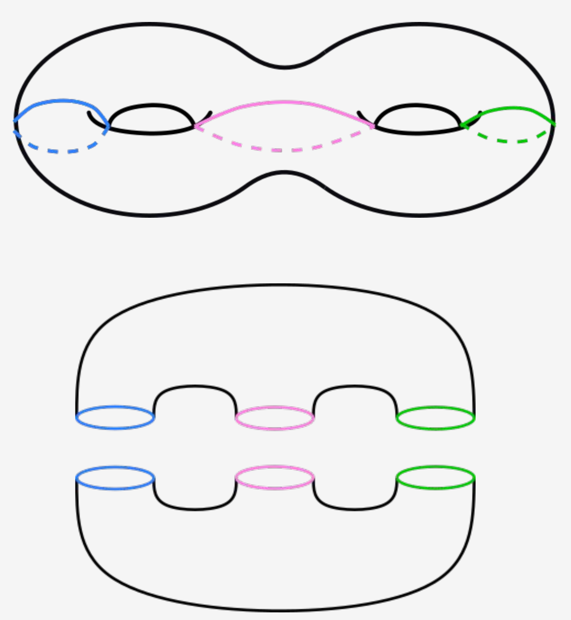
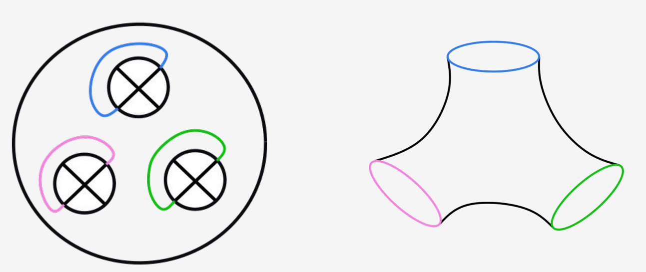
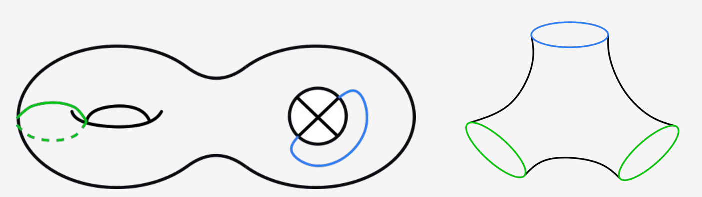
Curves Through Cross-Caps

Curves passing through a cross-cap can be disjoint through the entirety of the cross-cap.



Maximal Collections of Curves

A collection of disjoint curves is **maximal** if there does not exist another curve in the surface disjoint from the collection.

Orientable	vs. Non-Orientable
The number curves in a maximal collection depends only on genus.	The number curves in a maximal collection is NOT unique.
 Maximal collection of curves on genus 2 surface with 3 curves	 Maximal collection in N_3 with 3 curves  Maximal collection in N_3 with 2 curves

Theorem

Orientable Case [Malestein, Rivin, and Theran]

In a genus *g* surface, the maximum number of curves intersecting at most once is greater than or equal to

$$g^2 + \frac{5}{2}g.$$

Non-Orientable Case

In N_g the maximum number of curves intersecting at most once is greater than or equal to

$$\begin{cases} g^2 + \frac{9}{2}g + 2 & \text{g is even} \\ g^2 + \frac{5}{2}g + 2\left\lfloor \frac{g}{2} \right\rfloor + 1 & \text{g is odd.} \end{cases}$$

Acknowledgments

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