



THE BRAID GROUPS AND THEIR REPRESENTATIONS

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Knot at Lunch, 2021

THE BRAID GROUP

The **Braid group** on n strands, denoted B_n , has the following presentation:

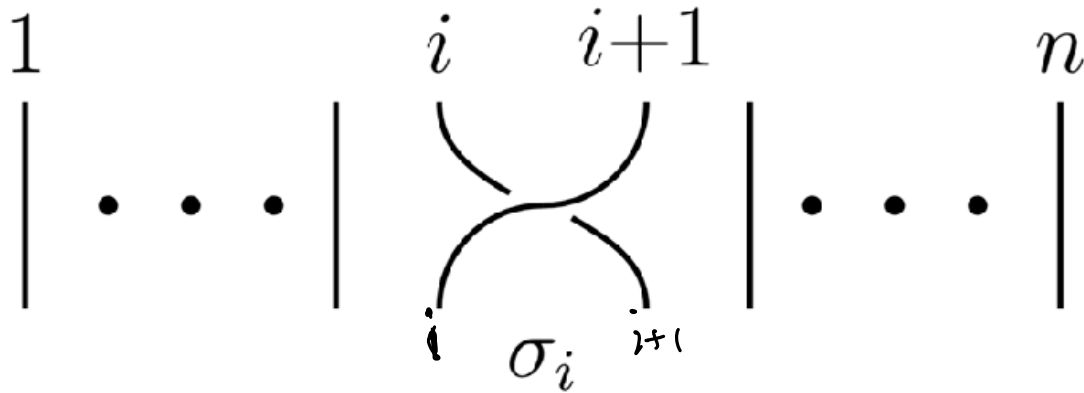
Generators: $\sigma_1, \dots, \sigma_{n-1}$

Relations: $\sigma_i \sigma_j = \sigma_j \sigma_i \quad j \neq i \pm 1$

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$$

VISUALIZING B_n

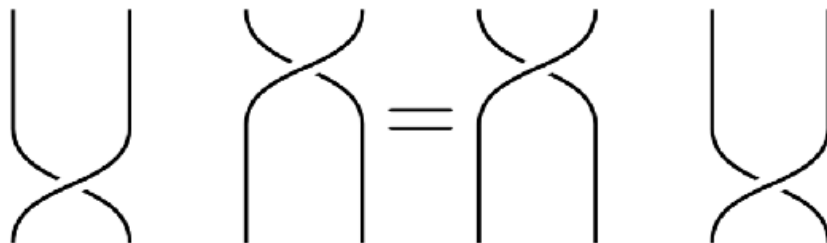
The generators σ_i



VISUALIZING B_n

Far commutativity relation

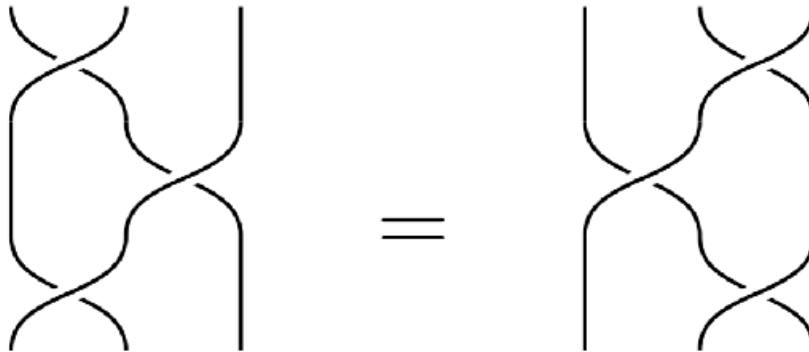
$$\sigma_i \sigma_j = \sigma_j \sigma_i \quad |i - j| > 1$$



VISUALIZING B_n

The braid relation

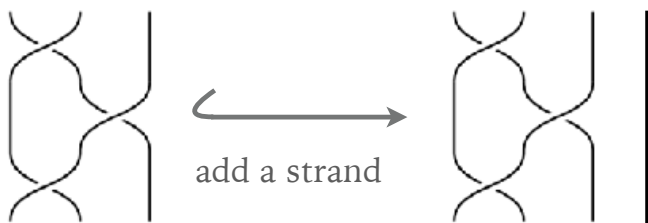
$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$$



FUN FACTS ABOUT B_n

➤ $|B_n| = \infty$

➤ $B_{n-1} \subseteq B_n$



$\sigma_1, \dots, \sigma_{n-2}, \sigma_{n-1}$

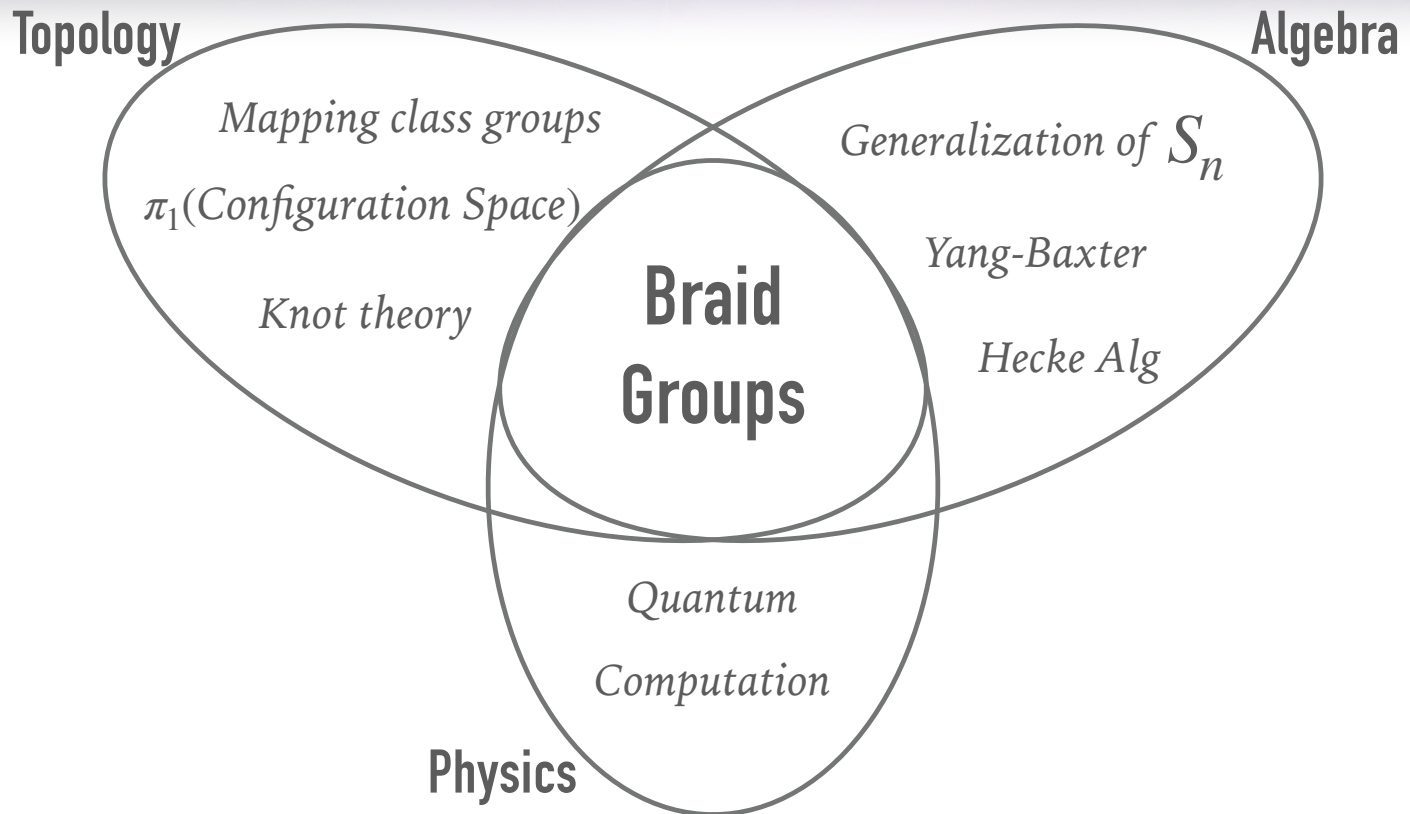
➤ $Z(B_n) = \langle (\sigma_1 \cdots \sigma_{n-1})^n \rangle$



➤ B_n is torsion free.

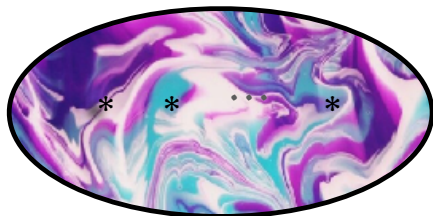
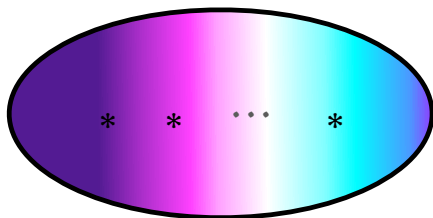
(No elements of finite order)

WHY SHOULD YOU CARE ABOUT B_n ?



B_n IN TOPOLOGY

Mapping Class Group of the
n-punctured disk

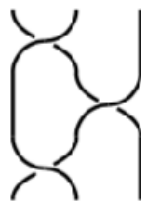


Fundamental Group of the
Configuration space

$$C_n = \{(x_1, \dots, x_n) \in \mathbb{C}^n \mid x_i \neq x_j \text{ if } i \neq j\}$$

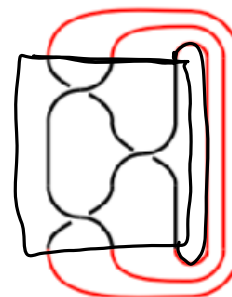
$$B_n \cong \pi_1(C_n/S_n)$$

Knot Theory



braid closure

*alexander's
theorem*



REPRESENTATIONS OF B_n

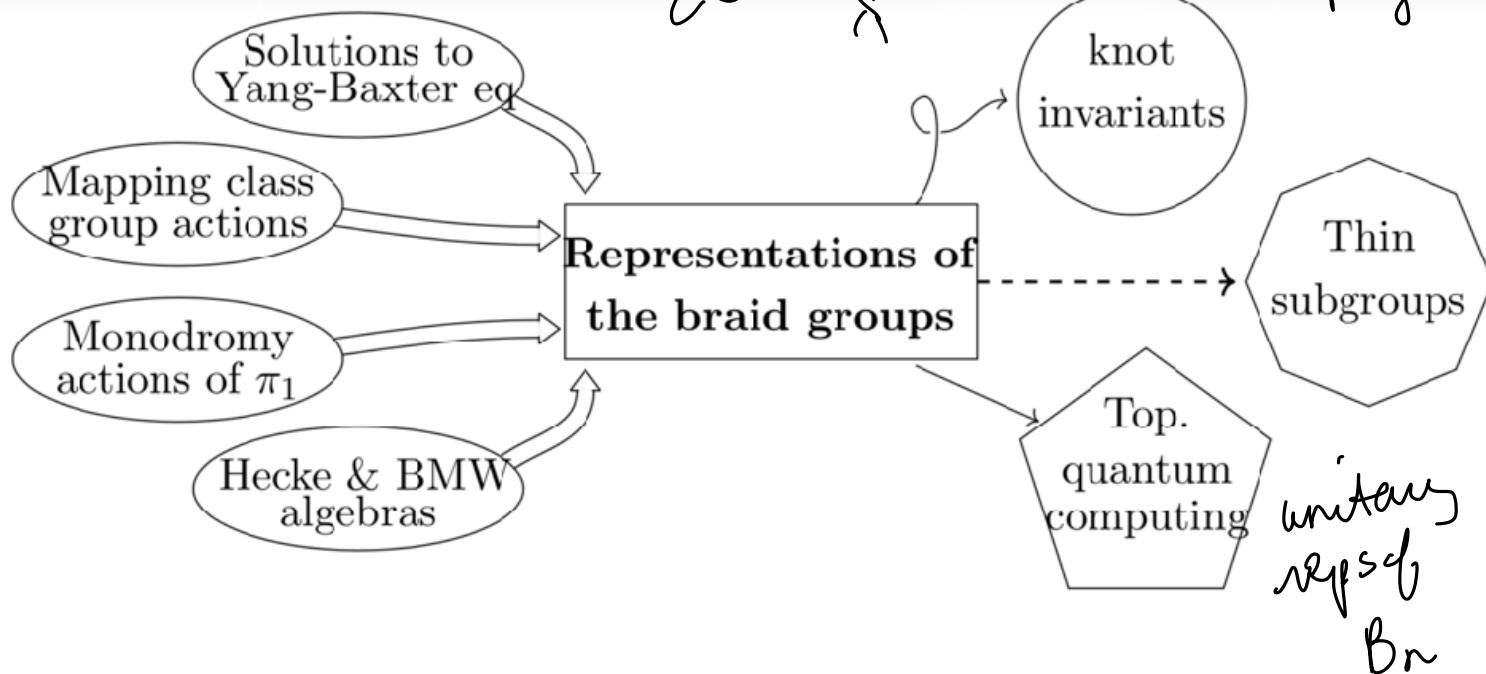
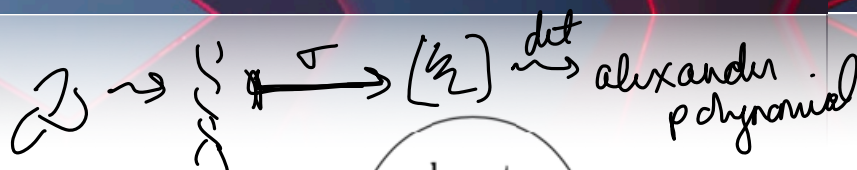
group homomorphisms

$$B_n \longrightarrow GL(V)$$

algebra homomorphisms

$$\mathbb{C}[B_n] \longrightarrow M_k(R)$$

REPRESENTATIONS OF B_n



THE BURAU REPRESENTATION

The (*reduced*) Burau representation $\rho_n : B_n \longrightarrow GL_{n-1}(\mathbb{Z}[t^{\pm 1}])$

$$\sigma_1 \mapsto \left(\begin{array}{cc|c} -t & 1 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & Id_{n-3} \end{array} \right), \sigma_{n-1} \mapsto \left(\begin{array}{c|cc} Id_{n-3} & 0 & 0 \\ \hline 0 & 1 & 0 \\ 0 & t & -t \end{array} \right)$$

$$\sigma_i \mapsto \left(\begin{array}{c|cccc} Id_{i-2} & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 \\ 0 & t & -t & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & Id_{n-i-2} \end{array} \right) \text{ for } 2 \leq i \leq n-2$$

THE BURAU REPRESENTATION

The (*reduced*) Burau representation $\rho_{n,t} : B_n \longrightarrow GL_{n-1}(\mathbb{Z}[t^{\pm 1}])$

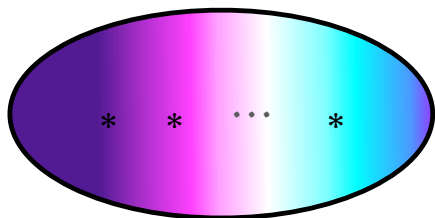
Need: An action of B_n on a $\mathbb{Z}[t^{\pm 1}]$ -module

Or

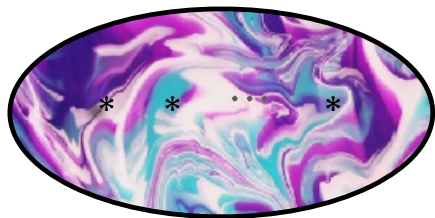
A way to view braids as iso's of a $\mathbb{Z}[t^{\pm 1}]$ -module

THE BURAU REPRESENTATION

$$\begin{array}{c} \hat{D}_n \\ \downarrow p \\ D_n \end{array}$$



$$\beta \in B_n$$

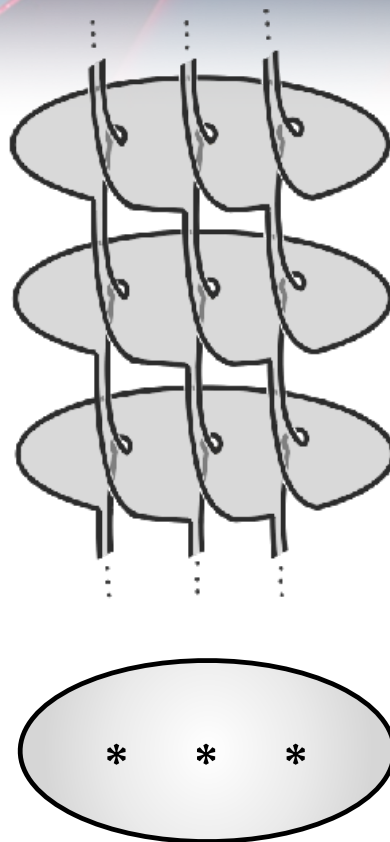


$$\beta : D_n \xrightarrow{\text{homeo}} D_n$$

THE BURAU REPRESENTATION

$$\hat{D}_n \xrightarrow{p} D_n$$

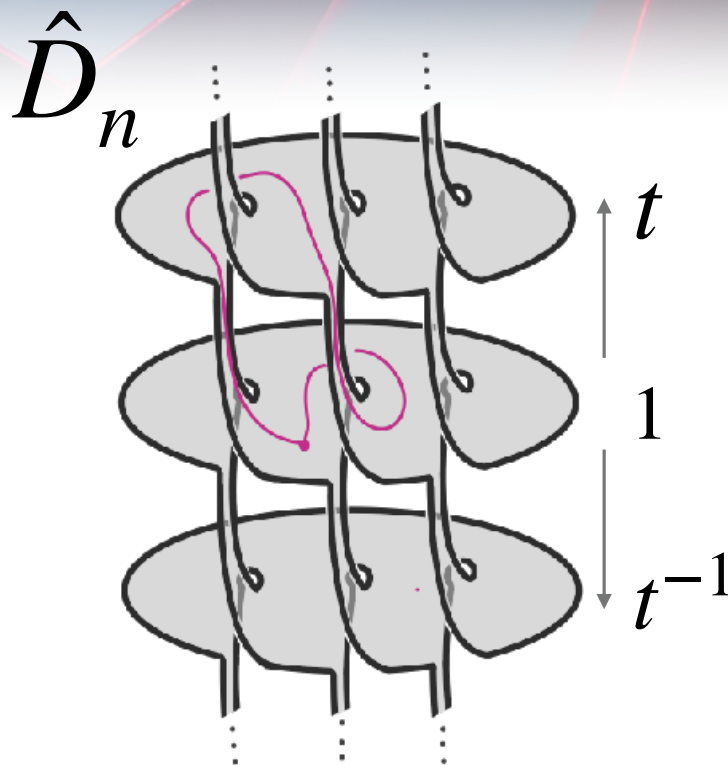
$$\begin{array}{c} \hat{D}_n \\ \downarrow p \\ D_n \end{array} \quad \text{Infinite cyclic cover}$$



THE BURAU REPRESENTATION

$$\begin{array}{c} \hat{D}_n \\ \downarrow p \\ D_n \end{array}$$

$H_1(\hat{D}_n)$ is a free
 $\mathbb{Z}[t^{\pm 1}]$ -module

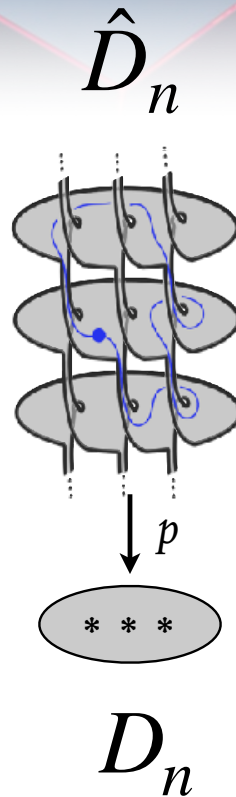


THE BURAU REPRESENTATION

$$\hat{D}_n \xrightarrow{p} D_n$$

$$\beta \in B_n$$

$$\beta : D_n \xrightarrow{\text{homeo}} D_n$$



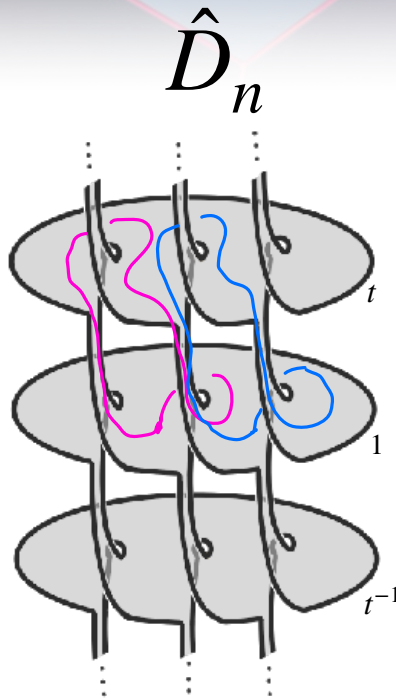
$$\hat{\beta}_* : H_1(\hat{D}_n) \xrightarrow{\text{iso}} H_1(\hat{D}_n)$$

$$\begin{array}{ccccc} & & \hat{D}_n & & \\ & \searrow^{\hat{\beta}} & \downarrow p & & \\ \hat{D}_n & \xrightarrow{p} & D_n & \xrightarrow{\beta} & D_n \end{array}$$

Diagram illustrating the relationship between $\hat{D}_n$, D_n , and $\hat{D}_n$ via maps p and β . A dashed arrow labeled $\hat{\beta}$ and a solid arrow labeled homeo connect the two $\hat{D}_n$ terms.

THE BURAU REPRESENTATION

$H_1(\hat{D}_n)$ is a free
 $\mathbb{Z}[t^{\pm 1}]$ -module



$$\hat{\beta}_* : H_1(\hat{D}_n) \xrightarrow{\text{iso}} H_1(\hat{D}_n)$$

$$\hat{\beta}_* \in Gl_{n-1}(\mathbb{Z}[t^{\pm 1}])$$

$$\rho : B_n \rightarrow Gl_{n-1}(\mathbb{Z}[t^{\pm 1}])$$

$$\beta \mapsto \hat{\beta}_*$$

THE BURAU REPRESENTATION

The (reduced) Burau representation $\rho_n : B_n \xrightarrow{|-|?} GL_{n-1}(\mathbb{Z}[t^{\pm 1}])$
 faithful? yes if $n \leq 3$ no if $n \geq 5$ $n=4$? open question!

$$\sigma_1 \mapsto \left(\begin{array}{cc|c} -t & 1 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & Id_{n-3} \end{array} \right), \sigma_{n-1} \mapsto \left(\begin{array}{c|cc} Id_{n-3} & 0 & 0 \\ \hline 0 & 1 & 0 \\ 0 & t & -t \end{array} \right)$$

$$\sigma_i \mapsto \left(\begin{array}{c|cccc} Id_{i-2} & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 \\ 0 & t & -t & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & Id_{n-i-2} \end{array} \right) \text{ for } 2 \leq i \leq n-2$$

Fun Fact :

The Burau representation extends to the virtual braid groups!

$$\begin{array}{ccc} & \textcolor{violet}{vB}_n & \\ \text{inclusion} \nearrow & & \searrow \rho_n \text{ Burau} \\ \textcolor{blue}{B}_n & \xrightarrow[\text{Burau}]{\rho_n} & GL_{n-1}(\mathbb{Z}[t^{\pm 1}]) \end{array}$$

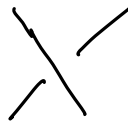
THE VIRTUAL BRAID GROUP

$$vB_n = B_n * S_n / \sim$$

vB_n

Intuitive definition (but less good—you'll see why soon)

B_n



S_n



Generators:

$$\sigma_1, \dots, \sigma_{n-1}$$

$$\tau_1, \dots, \tau_{n-1}$$

$$\tau_i = (i, i+1)$$

Relations:

$$\left. \begin{aligned} \sigma_i \sigma_j &= \sigma_j \sigma_i \quad j \neq i \pm 1 \\ \sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1} \end{aligned} \right\}$$

$$\left. \begin{aligned} \tau_i \tau_j &= \tau_j \tau_i \quad j \neq i \pm 1 \\ \tau_i \tau_{i+1} \tau_i &= \tau_{i+1} \tau_i \tau_{i+1} \\ \tau_i^2 &= e \end{aligned} \right\}$$

$$\sigma_i \tau_j = \tau_j \sigma_i \quad j \neq i \pm 1$$

$$\tau_i \sigma_{i+1} \tau_i = \tau_{i+1} \sigma_i \tau_{i+1}$$

Mixed
R-moves

THE VIRTUAL BRAID GROUP

B_n



∞

no torsion

topological

vB_n



∞

has finite order elements
 $\underline{S_n} (\times)^n = \mathbb{Q} = 11$

algebraic

THE VIRTUAL BRAID GROUP

B_n

$$B_n \xrightarrow{\pi} S_n$$

$$\sigma_i \mapsto \tau_i = (i, i+1)$$

$P_n = \ker(\pi)$ pure braid group

$$1 \rightarrow P_n \rightarrow B_n \rightarrow S_n \rightarrow 1$$

short exact seq

NOT True
 $\Rightarrow B_n \not\cong P_n \rtimes S_n$

vB_n

$$vB_n \xrightarrow{\pi} S_n$$

$$\sigma_i \mapsto \tau_i$$

$$\tau_i \mapsto \tau_i$$

$vPB_n = \ker(\pi)$ virtual pure braid group

$$1 \rightarrow vPB_n \rightarrow vB_n \rightarrow S_n \rightarrow 1$$

short exact seq
splits!!

$$vB_n = vPB_n \rtimes S_n$$

THE VIRTUAL BRAID GROUP

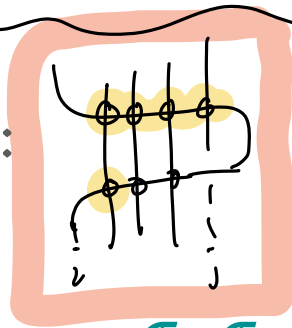
less good $vB_n = B_n * S_n / \sim$

$$\underline{vB_n} = \boxed{vPB_n} \rtimes \underline{S_n}_{\text{finite gp}}$$

THE **PURE** VIRTUAL BRAID GROUP

$$PvB_n = vPB_n = vP_n$$

Generators:



$\sigma_{i,j}$

$j \neq i, \quad 1 \leq i, j \leq n$



Relations:

$$\sigma_{i,j}\sigma_{k,l} = \sigma_{k,l}\sigma_{i,j} \quad i \neq j \neq k \neq l$$

$$\sigma_{i,j}\sigma_{i,k}\sigma_{j,k} = \sigma_{j,k}\sigma_{i,k}\sigma_{i,j}$$

WHEN DO REPRESENTATIONS “EXTEND”?

Given: $B_n \xrightarrow{\phi} GL_m(V)$

Does there exist?: $vB_n \xrightarrow{\hat{\phi}} GL_m(V)$
 So that $\hat{\phi}|_{B_n} = \phi$

But really we seek only this:

$B_n \not\subset vPB_n \xrightarrow{\hat{\phi}} GL_m(V)$

WHEN DO REPRESENTATIONS “EXTEND”?

Burau extends!

(unreduced)

Let V be a $\mathbb{Z}[t^{\pm 1}]$ -module with basis $v_1, \dots, v_n$

vPB_n acts on V by

$$\sigma_{i,j}(v_k) = \begin{cases} v_k & \text{if } k \neq j \\ tv_j - (t-1)v_i & \text{if } k = j \end{cases}$$

WHEN DO REPRESENTATIONS “EXTEND”?

Burau extends!

(unreduced)

Let V be a $\mathbb{Z}[t^{\pm 1}]$ -module with basis $v_1, \dots, v_n$

vPB_n acts on V by

$$\sigma_{i,j}(v_k) = \begin{cases} v_k & \text{if } k \neq j \\ tv_j - (t-1)v_i & \text{if } k = j \end{cases}$$

Gassner extends!

Let V be a $\mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ -module with basis $v_1, \dots, v_n$

vPB_n acts on V by

$$\sigma_{i,j}(v_k) = \begin{cases} v_k & \text{if } k \neq j \\ t_i v_j - (t_i - 1)v_i & \text{if } k = j \end{cases}$$

WHEN DO REPRESENTATIONS “EXTEND”?

What about Lawrence-Krammer-Bigelow? *it's faithful!!*

Let V be a $\mathbb{Z}[t^{\pm 1}, q^{\pm 1}]$ -module with basis $v_1, \dots, v_k$, where $k = \frac{n(n-1)}{2}$

B_n acts on V by

$$\sigma_i \cdot v_{j,k} = \begin{cases} v_{j,k} & i \notin \{j-1, j, k-1, k\}, \\ qv_{i,k} + (q^2 - q)v_{i,j} + (1 - q)v_{j,k} & i = j - 1 \\ v_{j+1,k} & i = j \neq k - 1, \\ qv_{j,i} + (1 - q)v_{j,k} - (q^2 - q)tv_{i,k} & i = k - 1 \neq j, \\ v_{j,k+1} & i = k, \\ -tq^2v_{j,k} & i = j = k - 1. \end{cases}$$



THANK YOU!