From Tangles to Links

A story of knots and Gaußian Differential Operators

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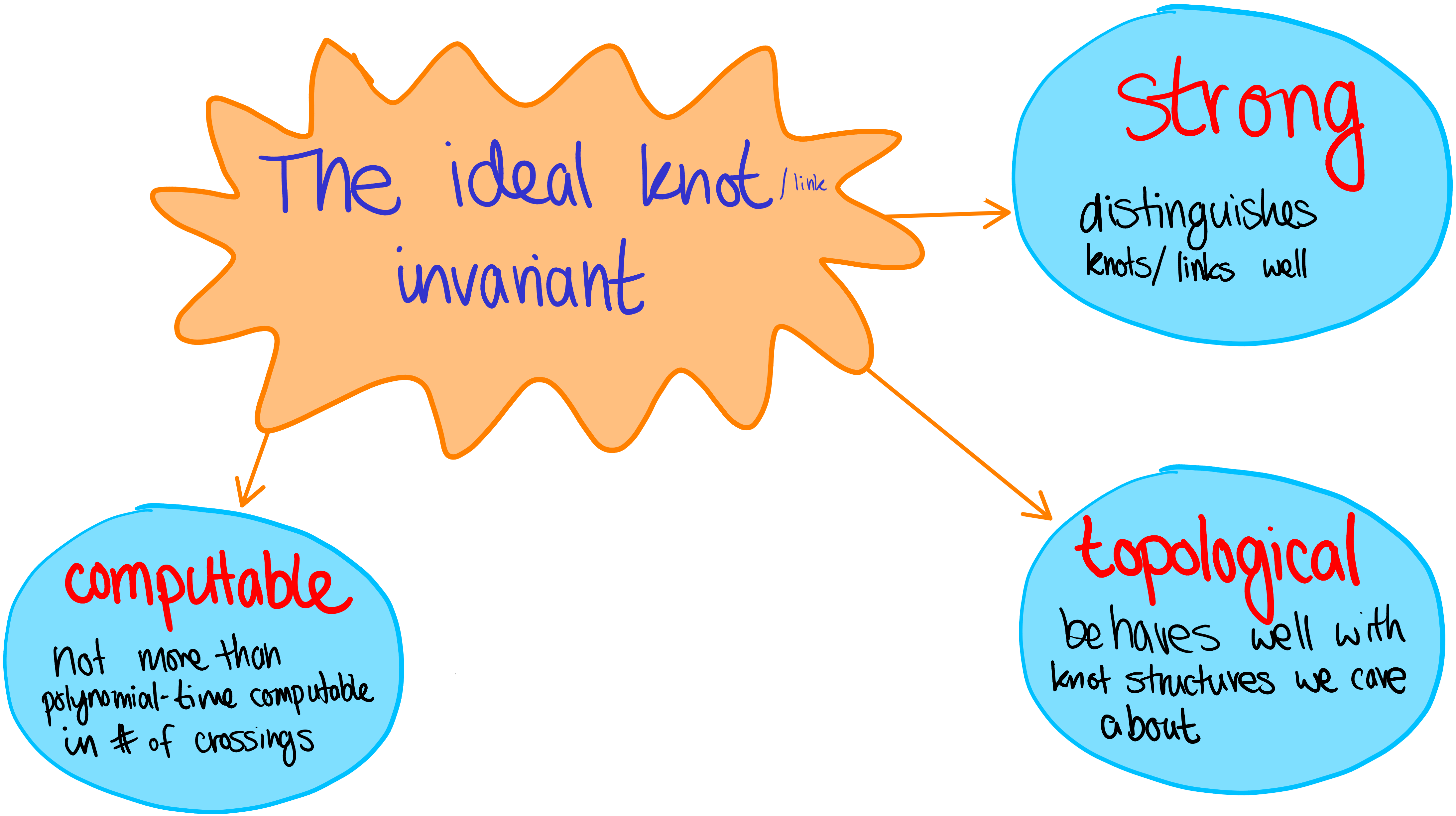

The ideal knot / link invariant
strong
distinguishes knots/ links well
computable
not more than polynomial-time computable in # of crossings
topological
behaves well with knot structures we care about

Hopf Algebras

Mat

$\mathbb{Q} \to$ diag

(1) algebra structure:

- vector space A and tensor products $A\otimes A$, $A\otimes A\otimes A, \dots$
- multiplication $A\otimes A \xrightarrow{m} A$
- unit a map $\Bbbk \xrightarrow{\eta} A$ (define $\mathbb{1} := \eta(1) \in A$)

$\mathbb{Q}$

$\mathbb{1} \in A$ $\quad \forall a \in A$

$\boxed{\mathbb{1}\cdot a = a \cdot \mathbb{1} = a}$

$\eta(1) =: \mathbb{1}$

$\eta(r) = r\,\eta(1) = r\cdot\mathbb{1}$

- relations:

$$\begin{tikzcd}
A\otimes A\otimes A \arrow[r, "m\otimes id"] \arrow[d, "id\otimes m"'] \arrow[dr, phantom, "\circlearrowleft"] & A\otimes A \arrow[d, "m"] \\
A\otimes A \arrow[r, "m"'] & A
\end{tikzcd}$$

$(ab)c$

$\shortparallel$

$a(bc)$

associativity

$$\begin{tikzcd}
A \arrow[r, "id\otimes\eta"] \arrow[d, "\eta\otimes id"'] \arrow[dr, "id"] & A\otimes A \arrow[d, "m"] \\
A\otimes A \arrow[r, "m"'] & A
\end{tikzcd}$$

unit

Hopf Algebras

② coalgebra structure:

- vector space A and tensor products $A\otimes A$, $A\otimes A\otimes A, \dots$

- comultiplication $A\otimes A \xleftarrow{\Delta} A$ "diagonal" "doubling" "factorize"

- counit a map $\Bbbk \xleftarrow{\varepsilon} A$ } trivial rep.

- relations:

$$\begin{tikzcd}
A\otimes A\otimes A & A\otimes A \arrow[l, "\Delta\otimes id"'] \\
A\otimes A \arrow[u, "id\otimes\Delta"] & A \arrow[l, "\Delta"] \arrow[u, "\Delta"'] \arrow[ul, phantom, "\circlearrowleft"]
\end{tikzcd}$$

coassociativity

$A\otimes\Bbbk$

$$\begin{tikzcd}
\Bbbk\otimes A & A\otimes A \arrow[l, "id\otimes\varepsilon"'] \\
A\otimes A \arrow[u, "\varepsilon\otimes id"] & A \arrow[l, "\Delta"] \arrow[u, "\Delta"'] \arrow[ul, "id"']
\end{tikzcd}$$

counit

Hopf Algebras

③ bialgebra structure:

- vector space A and tensor products $A\otimes A$, $A\otimes A\otimes A, \ldots$
- algebra structure m, η
- coalgebra structure Δ, ε

- antipode ("inverse") $S : A \to A$

$(a\otimes b)\cdot(c\otimes d) = ac \otimes bd$

$\Delta : A \longrightarrow A\otimes A$

- relations

$$\begin{tikzcd}
 & A\otimes A \arrow[r, "id\otimes S"] & A\otimes A \arrow[dr, "m"] & \\
A \arrow[ur, "\Delta"] \arrow[r, "\varepsilon"'] & \Bbbk \arrow[rr, "\eta"'] \arrow[u, phantom, "\circlearrowright"] & & A
\end{tikzcd}$$

$g \mapsto g\otimes g \mapsto g\otimes g^{-1} \mapsto \mathbb{1}$

$g \mapsto 1 \mapsto \mathbb{1}$

antipode relation

("$gg^{-1} = \mathbb{1}$")

$$\begin{tikzcd}
A\otimes A \arrow[r, "\Delta\otimes\Delta"] \arrow[dd, "m"'] & (A\otimes A)\otimes(A\otimes A) \arrow[d, "id\otimes\tau\otimes id"] \\
\arrow[r, phantom, "\circlearrowright"] & (A\otimes A)\otimes(A\otimes A) \arrow[d, "m\otimes m"] \\
A \arrow[r, "\Delta"'] & A\otimes A
\end{tikzcd}$$

$\boxed{\tau(x\otimes y) = y\otimes x}$

Δ is an algebra morphism/
m is a coalgebra morphism

$$\boxed{\Delta(xy) = \Delta(x)\Delta(y)}$$

Hopf Algebras

Examples

Group algebras

G a group, $\Bbbk$ a field

$A = \Bbbk[G]$

$m(g \otimes h) = gh$

$\eta(\lambda) = \lambda . e$ $\quad (\mathbb{1} = e)$

$\Delta(g) = g \otimes g$

$\varepsilon(g) = 1$ for $g \in G$

$S(g) = g^{-1}$ — " —

Lie Algebras

$\mathfrak{g}$ Lie alg. $= \operatorname{span}\{X, Y, \ldots\}$

$A = \underset{\hbar}{\widehat{U}}(\mathfrak{g}) =$ words in $\mathfrak{g}$

$\Bbbk\langle X, Y, \ldots \rangle / [X,Y] = XY - YX$

$m(X \otimes Y) = XY$

$\Delta(X) = X \otimes \mathbb{1} + \mathbb{1} \otimes X$

$\varepsilon(X) = 0$ for $X \in \mathfrak{g}$

$S(X) = -X$ — " —

$G \circlearrowright V, W$

$G \circlearrowright V \otimes W$

$g \cdot (v \otimes w) = g.v \otimes g.w$

$= (g \otimes g) . (v \otimes w)$

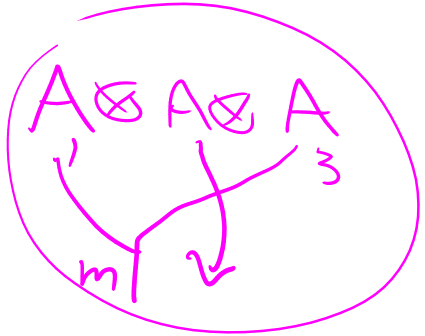
$= (\Delta g) . (v \otimes w)$

$G \circlearrowright \Bbbk$

$g . r = r$

$X . (v \otimes w) = (X . v) \otimes w + v \otimes X . w$

Meta-Hopf Algebras



Label tensor factors with elements of a finite set:

Instead of $\overset{\{1,\,2,\,3\}}{A\otimes A\otimes A} = A^{\otimes 3}$ $\rightsquigarrow$ $A_i\otimes A_j\otimes A_k \doteq A^{\otimes\{i,j,k\}}$

Instead of $m : A\otimes A \to A$ $\rightsquigarrow$ $m^{ij}_k : A^{\otimes\{i,j\}} \to A^{\otimes\{k\}}$

$A^{\otimes 0} := k$

Instead of a v.s. A and tensors $A^{\otimes 2}, A^{\otimes 3}, \dots$ $\rightsquigarrow$ For S a finite set, assign A_S a set.

$(\underbrace{A, A^{\otimes n}}, \underbrace{m, \Delta, \eta, \varepsilon, S})$

$$\boxed{(\{A_S\}_{S \text{ fin set}}, m^{ij}_k, \Delta^i_{jk}, \eta_k, \varepsilon^i, S^i_i)}$$

$$m^{ij}_k : A^{\otimes\{i,j\}\cup S} \longrightarrow A^{\otimes\{k\}\cup S}$$

$$\eta_k : A^{\otimes S} \longrightarrow A^{\otimes S\cup\{k\}}$$

$\eta_k : A_S \to A_{S\cup\{k\}}$, $\eta_k : A_\emptyset \to A_{\{k\}}$

Meta- Hopf Algebras

① meta- algebra structure:

- Set A and ~~tensor products~~ associated sets $\cancel{A\otimes A}$, $\cancel{A\otimes A\otimes A}$, ... $A_{\{i,j\}}$, $A_{\{i,j,k\}}$, A_S for $|S|<\infty$

- meta- multiplication $A_{\{i,j\}} \xrightarrow{m^{ij}_k} A_{\{k\}}$

- meta- unit a map $A_\phi \xrightarrow{\eta_i} A_{\{i\}}$ ~~(define $1\!\!1 := \eta(1) \in A$)~~

- relations:

$$\begin{tikzcd}
A_{\{i,j,k\}} \arrow[r, "m^{ij}_r"] \arrow[d, "m^{jk}_s"'] \arrow[dr, phantom, "\circlearrowright"] & A_{\{r,k\}} \arrow[d, "m^{rk}_t"] \\
A_{\{i,s\}} \arrow[r, "m^{is}_t"'] & A_{\{t\}}
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
A_{\{j\}} \arrow[r, "\eta_i"] \arrow[d, "\eta_k"'] \arrow[dr, "id"] & A_{\{i,j\}} \arrow[d, "m^{ij}_j"] \\
A_{\{j,k\}} \arrow[r, "m^{jk}_j"'] & A_{\{j\}}
\end{tikzcd}$$

meta- associativity $\qquad\qquad$ meta- unit

Meta-Hopf Algebras

② meta-co algebra structure:

- Set A and ~~tensor products~~ associated sets ~~$A\otimes A$, $A\otimes A\otimes A, \dots$~~ $A_{\{i,j\}}$, $A_{\{i,j,k\}}$, A_S for $|S|<\infty$

- meta-co multiplication $A_{\{j,k\}} \xleftarrow{\Delta^i_{jk}} A_{\{i\}}$

- meta-co unit a map $A_\phi \xleftarrow{\varepsilon^i} A_{\{i\}}$

- relations:

$$\begin{tikzcd}
A_{\{i,j,k\}} & A_{\{r,k\}} \arrow[l, "\Delta^r_{ij}"'] \\
A_{\{i,s\}} \arrow[u, "\Delta^s_{jk}"] \arrow[ur, phantom, "\circlearrowright"] & A_{\{t\}} \arrow[l, "\Delta^t_{is}"] \arrow[u, "\Delta^t_{rk}"']
\end{tikzcd}$$

meta-co associativity

$$\begin{tikzcd}
A_{\{j\}} & A_{\{j,k\}} \arrow[l, "\varepsilon_k"'] \\
A_{\{i,j\}} \arrow[u, "\varepsilon_i"] & A_{\{j\}} \arrow[l, "\Delta^j_{ij}"] \arrow[u, "\Delta^j_{jk}"'] \arrow[ul, "\mathrm{id}"']
\end{tikzcd}$$

meta-co unit

Meta-Hopf Algebras

$|S| = |T|$

$A_S \cong A_T$

(3) meta-bialgebra structure:

- ~~vector space~~ set $A_{\{i\}}$ and ~~tensor products~~ sets $A \otimes A$, $A \otimes A \otimes A, \ldots$
- meta-algebra structure m^{ij}_k, η_i
- meta-coalgebra structure $\Delta^i_{jk}, \varepsilon^i$
- meta-antipode ("inverse") $S^i_j : A_{\{i\}} \to A_{\{j\}}$
- relations

$$\begin{tikzcd}
& A_{\{j,k\}} \arrow[r, "S^k_k"] \arrow[d, phantom, "\circlearrowleft"] & A_{\{j,k\}} \arrow[dr, "m^{jk}_i"] & \\
A_{\{i\}} \arrow[ur, "\Delta^i_{jk}"] \arrow[r, "\varepsilon^i"'] & A_\emptyset \arrow[rr, "\eta_i"'] & & A_{\{i\}}
\end{tikzcd}$$

meta-antipode relation

("$gg^{-1} = 1$")

$$\begin{tikzcd}
A_{\{i,j\}} \arrow[r, "\Delta^i_{ab} \circ \Delta^j_{cd}"] \arrow[d, "m^{ij}_k"'] \arrow[dr, phantom, "\circlearrowleft"] & A_{\{a,b,c,d\}} \arrow[d, "m^{ac}_i \circ m^{bd}_j"] \\
A_{\{k\}} \arrow[r, "\Delta^k_{ij}"'] & A_{\{i,j\}}
\end{tikzcd}$$

Δ is a meta-algebra morphism /
m is a meta-coalgebra morphism

$\boxed{\cancel{\tau(x \otimes y) = y \otimes x}}$

unnecessary!

Meta-Hopf Algebras

Examples

Group algebras

G a group, $\Bbbk$ a field

$A_S = \Bbbk[G]^{\otimes S}$

$m^{ij}_k(g_i h_j) = g_k h_k$

$\eta_i(\lambda) = \lambda . e_i$

$\Delta^i_{jk}(g_i) = g_j g_k$

$\varepsilon^i(g_i) = 1$ for $g \in G$

$S^i_i(g_i) = g_i^{-1}$ —"—

Lie Algebras

$\mathfrak{g}$ Lie alg.

$A_S = \widehat{U_\hbar(\mathfrak{g})}^{\otimes S}$

$m^{ij}_k(X_i Y_j) = X_k Y_k$

$\Delta^i_{jk}(X_i) = X_j + X_k$

$\varepsilon^i(X_i) = 0$ for $X \in \mathfrak{g}$

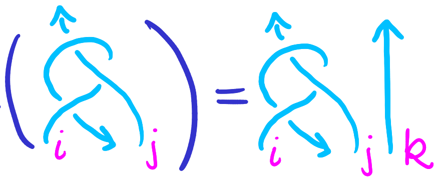
$S^i_i(X_i) = -X_i$ —"—

$\in A_{\{i,j\}}$

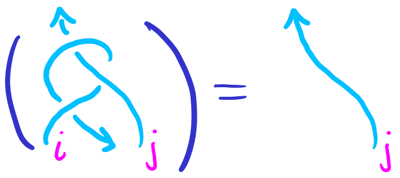
(virtual) tangles (vT)

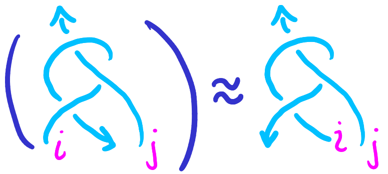
$A_S = \{$open tangles with components indexed by $S\}$

m^{ij}_k 

η_k 

Δ^i_{ab} 

ε^i 

S^i_i 

Meta-Hopf Algebras

Examples

The important algebra example:

$$\mathfrak{g} = \langle y, b, a, x \rangle \Big/ \begin{array}{l} [a,x]=x \\ [a,y]=-y \\ [x,y]=b \\ b \text{ is central} \end{array}$$

$$\overset{i}{y} \otimes \overset{j}{x} =: y_i x_j$$

Let $\mathcal{U} = \mathcal{U}(\mathfrak{g})[[\hbar]] = \widehat{\mathcal{U}(\mathfrak{g})}$ = power series in y, b, a, x

$\mathcal{U}_S = \mathcal{U}^{\otimes S} = \langle y_i, b_i, a_i, x_i \rangle_{i \in S} \Big/$ (different indices commute)

e.g.

$$\boxed{i \neq j \Rightarrow x_i y_j = y_j x_i}$$

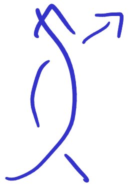
Thm There is a morphism of meta-Hopf algebras:

$$Z: vT \longrightarrow U$$

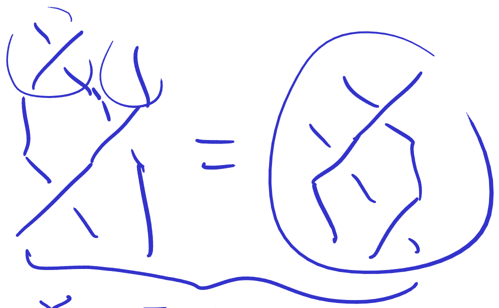
 $\longmapsto e^{a_j b_i + y_i x_j} =: R_{ij}$

 $\longmapsto e^{-a_j b_i - y_i x_j e^{b_i}} =: \bar{R}_{ij}$

which is also invariant.



$$R_{ij}^{-1} = \bar{R}_{ij}$$

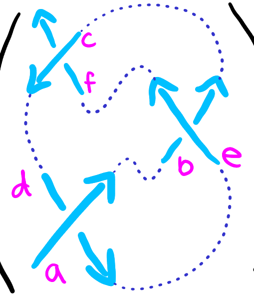


Yang-Baxter equation for R_{ij}

$\Delta \longrightarrow \delta$

This is topological, but is it efficient?

Computational optimization

$Z\Big($  $\Big) = Z\Big($  $\Big)$

$$= \underbrace{\overline{R}_{ad}\,\overline{R}_{eb}\,\overline{R}_{cf}}_{\in\, \mathcal{U}^{\otimes\{a,b,c,d,e,f\}}} \,/\!/\, m_i^{ab} \,/\!/\, m_i^{ic} \,/\!/\, m_i^{id} \,/\!/\, m_i^{ie} \,/\!/\, m_i^{if}$$

The bracket relations $[a,x]=x, \ldots$ allow us to choose an ordering of monomial factors.
$ax = xa + x$

$$\rightsquigarrow \ \mathcal{U}(\mathfrak{g}) \cong \mathbb{k}[\![y,b,a,x]\!] \text{ as vector spaces}$$

Q: How do we compute maps between power series rings quickly?

The hom-⊗ adjunction

Fact

Given vector spaces U, V, W

$$\hom(U \otimes V, W) \cong \hom(V, \hom(U, W))$$

We can identify linear maps $V \longrightarrow W$ with elements of $V^* \otimes W$

The map $m^{ij}_k : U^{\otimes\{i,j\}} \longrightarrow U^{\otimes\{k\}}$ can be re-interpreted as $m^{ij}_k \in (U^*)^{\otimes\{i,j\}} \otimes U^{\otimes\{k\}} =: U^{\otimes\{i^*, j^*, k\}}$

(exponential generating functions)

Fact (composing maps)

Given $\varphi : \mathbb{k}[X] \longrightarrow \mathbb{k}[Y]$ (i.e. $\varphi \in \mathbb{k}[\![X^*, Y]\!]$) and $f \in \mathbb{k}[X]$

$$ev_f(\varphi) = \varphi(f) = \left(\varphi\big|_{X^* = \frac{\partial}{\partial X}}\right)(f)\Big|_{X=0}$$

$$\langle (X^*)^n, X^m \rangle = \delta_{n,m}\, n! = \frac{\partial^n}{\partial X^n} X^m \Big|_{X=0}$$

Composing generating functions

In $\mathcal{U}(\mathfrak{g})$, $m^{ij}_k = e^{(y_i^* + e^{-a_i^*} y_j^*) y_k + (b_i^* + b_j^*) b_k + (a_i^* + a_j^* + y_j^* x_i^*) a_k + (e^{-a_j^*} x_i^* + x_j^*) x_k}$

Putting weights on the variables such that:

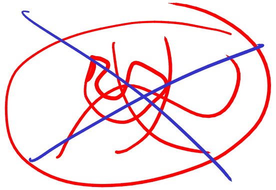
- $wt(c_i) + wt(c_i^*) = 2$ for all $c \in \{y, b, a, x\}$
- $wt(x_i) = wt(y_i) = 1$, $wt(b_i) = 0$, $wt(a_i) = 2$

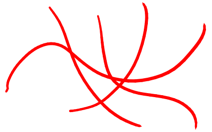
$\Rightarrow m^{ij}_k$ is of the form e^Q for a quadratic Q.

[more computation] $\Rightarrow \Delta^i_{jk}, \eta_i, \varepsilon^i, S^i_i, R_{ij}$ are all Gaussians!

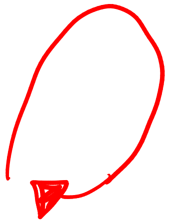
$\boxed{\text{Quadratics grow quadratically}}$

Traces

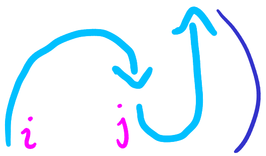
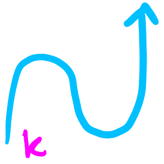
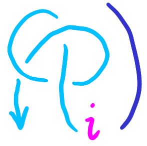
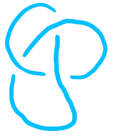




For matrices $\boxed{(AB)_{ik} = \sum_j a_{ij} b_{jk}}$ and $\boxed{tr(A) = \sum_i a_{ii}}$

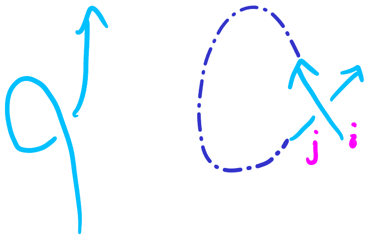


⤳ multiplication glues two matrices together
traces glue a matrix to itself

For knots m^{ij}_k () =  and ? () = 

tr^i

For a general algebra A, a trace on A is a coequalizer $A \otimes A \underset{m^{op}}{\overset{m}{\rightrightarrows}} A \xrightarrow{tr} \underbrace{A_A}_{\text{coinvariants of } A}$



$= R$ " m^{ij}_k